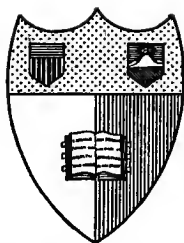




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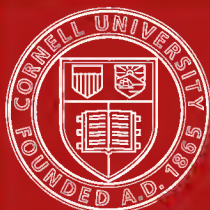
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# LESSONS IN MECHANICS

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# LESSONS IN MECHANICS

A TEXT-BOOK FOR COLLEGES AND  
TECHNICAL SCHOOLS

BY  
WILLIAM S. FRANKLIN AND BARRY MACNUTT

BETHLEHEM, PENNSYLVANIA  
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1919



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## PREFACE.

This book and a companion volume entitled *Lessons in Electricity and Magnetism* have been arranged to meet the needs of the two-year schedule in elementary physics which has been recently adopted in some of our technical schools, and, in our opinion, most of the added emphasis which this two-year schedule involves should be directed towards the mathematical side, or, let us say, the mathematical bottom of the subject.

The two-year schedule in elementary physics, beginning with the Freshman year, means that teachers of physics cannot base their work upon college courses in mathematics to the extent that might have been possible heretofore; but we believe, nevertheless, that teachers of physics can and should make use of the more powerful mathematical methods from the beginning. Accordingly we have used differential and integral calculus throughout these new texts, but without demanding the previous study of calculus by the student.

The central idea underlying these new texts is that mathematical training must be accomplished by the combined efforts of teachers of mathematics and teachers of physics, and it is hoped that these texts may help to bring our physics teachers to what we believe to be their proper function in the important and difficult matter of mathematical training. If our students were to come from their mathematics teachers fully able to use their mathematics there would be nothing left for the rest of us to do!

The central purpose of these new texts is to facilitate classroom work. Descriptive and explanatory material has been reduced to a minimum, because students will not read more than is absolutely necessary; the development of every topic leads

as directly as possible to illustrative numerical problems; and everything has been arranged in strict lesson order.

The authors are indebted to members of the staff of the Department of Physics of the Massachusetts Institute of Technology and to Lieut. Col. Fred M. Green, C. V., for many helpful suggestions.

THE AUTHORS.

## HELPFUL REFERENCES AND IMPORTANT BOOKS ON MECHANICS.

### (a) BOOKS AND REFERENCES FOR COLLATERAL READING BY THE BEGINNER.

*The Science of Mechanics*, Ernst Mach, translated from the German by Thomas J. McCormack, Chicago, The Open Court Publishing Co., 1893.  
*Properties of Matter*, Poynting and Thomson, London, Charles Griffen & Co., 1902. The general discussion of Gravitation, pages 7-52, the discussion of Elasticity pages 53-134, the discussion of Capillarity, pages 135-181, and the discussion of the Viscosity of Liquids, pages 205-223, are especially noteworthy.

*Textbook of Physics*, O. D. Chwolson, Vol. I (French translation).

### (b) BOOKS FOR ADDITIONAL STUDY (WITH EMPHASIS ON PHYSICS).

*Treatise on Dynamics* (being Vol. I of a *Treatise on Physics*), Andrew Gray.  
*Cours de Mechanique*, Bonasse, Paris, Libraire Ch. Delagrave.

*Handbuch der Theoretischen Physik* (Vol. I), Winkelmann.

*Aeronautics*, E. B. Wilson, John Wiley and Sons, New York, 1919.

*Spinning Tops and Gyroscopic Motion*, H. Crabtree, London, Longmans, Green and Co., 1909.

*Gyrostatics and Rotational Motion*, Andrew Gray, London, Macmillan & Co., 1918.

*Theorie Nouvelle de la Rotation des Corps*, Poinsot; an extremely interesting old book to be found in any good library.

*Theorie des Kreisels*, Klein and Sommerfeld, B. G. Teubner, Leipsig.

### (c) BOOKS FOR ADDITIONAL STUDY (WITH EMPHASIS ON ENGINEERING).

*Applied Mechanics* (2 vols.), Fuller and Johnston, John Wiley & Son, New York, 1914 and 1919.

The article *Hydromechanics* in the 9th edition of the Encyclopedia Britannica, by Unwin, is a very good engineering treatise on *Hydrostatics* and *Hydraulics*.

*Vorlesungen über Technische Mechanik*, 5 vols., A. Föppl, Leipzig, B. G. Teubner, 1898-1907.

### (d) BOOKS CHIEFLY ANALYTICAL.

*Treatise on Statics*, 2 vols., Geo. M. Minchin, Oxford, 1884.

*Analytical Statics*, 2 vols., E. J. Routh, Cambridge, 1896.

*Dynamics of a Particle (Dynamics of Translation)*, E. J. Routh, Cambridge, 1898.

*Rigid Dynamics, Elementary and Advanced (Dynamics of Rotation)*, 2 vols., E. J. Routh, London, Macmillan & Co., 1877 and 1892.

*Mathematical Theory of Elasticity*, A. E. H. Love, Cambridge, 1892.

*Treatise on Hydrodynamics*, Horace Lamb, Cambridge, 1916 (fourth edition).

## (e) BOOKS ON RELATIVITY.

*The Principle of Relativity*, E. Cunningham, Cambridge, 1914.

*Relativity and the Electron Theory*, E. Cunningham, Cambridge, 1915.

See also a very recent book on this subject by R. C. Tolman.

## (f) THE GREAT HISTORICAL WORKS ON MECHANICS.

Newton's *Principia*.

Lagrange's *Mechanique Analytique*.

Thomson and Tait's *Natural Philosophy*.

## (g) COLLECTIONS OF PHYSICAL AND CHEMICAL DATA.

*Tables Annuelles Internationales des Constants*.

*Physikalisch-chemische Tabellen*, Landolt-Börnstein.

*Smithsonian Tables* (Fowle); publication No. 2269.

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Every advanced student of physics and everyone who is interested in physical research should know about the *American Physical Society* and its official organ of publication, *The Physical Review*. The secretary of the society is Professor D. C. Miller of the Case School of Applied Science, Cleveland, Ohio, and the managing editor of the *Review* is Professor Frederick Bedell of Cornell University, Ithaca, N.Y.

## THE UNITED STATES BUREAU OF STANDARDS.\*

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\* The corresponding establishment in Great Britain is The National Physical Laboratory.

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All of these societies issue important publications, and all, or nearly all, of them make special provision for student members. A letter addressed to the headquarters office of any of the societies will bring full information concerning the society.

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## CHAPTER I.

### SIMPLE STATICS.\*

**1. What is meant by force?**—When one pushes or pulls on an object one is said to *exert a force* on the object. The pull of a horse on a wagon is a *force*, the pull of a locomotive on a train is a *force*. It does not, however, require an active agent like a horse or a locomotive to exert a force; thus a weight lying on a table exerts a downward push on the table (a *force*).

The gravity pull of the earth on a one-pound body is widely used as a unit of force and it is called a "*pound*." It is not the same everywhere, and, strictly speaking, the "*pound*" is defined as the pull of the earth on a one-pound body in London.

*When the word pound refers to an amount of material, as measured by a balance scale, it is a unit of mass; and it might be called the sugar-pound.*

*When the word pound refers to an amount of push or pull it is a unit of force, and when the word has this meaning it is set off in quotation marks throughout this text. The "pound" of force might very properly be called the pull-pound.*

Units of mass and force are discussed in Arts. 29 to 30 of Chapter II.

**Tension and compression.**—The rope in Fig. 1 is said to be in *tension*, and the tension of the rope is 50 "*pounds*," let us say, if the rope exerts a pull of 50 "*pounds*" at *a* and a pull of 50 "*pounds*" at *b*. The weight of the rope is neglected in this statement.

The column in Fig. 2 is said to be in *compression*, and the compression of the column is 20,000 "*pounds*," let us say, if the

\* This chapter does not deal with elastic forces nor with forces in fluids, that is to say, this chapter does not deal with the statics of elasticity nor with the statics of fluids (hydrostatics).

column exerts a push of 20,000 "pounds" at  $a$  and a push of 20,000 "pounds" at  $b$ . The weight of the column is neglected in this statement.

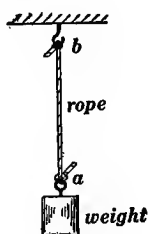


Fig. 1.

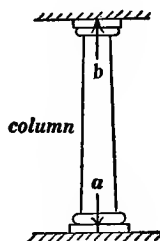


Fig. 2.

2. A set of forces which all act at the same point. Resultant

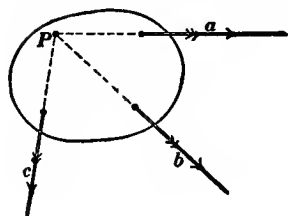


Fig. 3.

of the set.—Consider a number of forces which act on a given body. If the lines of action of all the forces intersect at a point  $P$  as indicated in Fig. 3 the forces are said to *act at a point*. Such a set of forces is always equivalent to a single force acting at  $P$ , and this single force

is called the *resultant* of the set.\*

(a) Resultant of two forces which act at a point. The addition triangle.—The two lines  $a$  and  $b$  in Fig. 4 represent

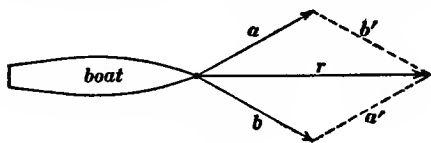


Fig. 4.



Fig. 5.

two forces which act on a boat. Completing the parallelogram  $aa'bb'$ , we get the diagonal  $r$  which represents the resultant of  $a$  and  $b$ .

\* If the lines of action of all the forces do not intersect at a point, the set of forces may or may not be equivalent to a single force. See Art. 13.

In a certain sense the resultant  $r$  is the *sum* of the two forces  $a$  and  $b$ , because  $r$  is exactly equivalent to  $a$  and  $b$  taken together; but  $r$  is by no means the sum of  $a$  and  $b$  in a simple arithmetical sense, it is the sum in an entirely new sense. It is called the *vector sum* of  $a$  and  $b$ .

*The addition triangle.*—The geometrical relation between  $a$ ,  $b$  and  $r$  in Fig. 4 is completely shown by the triangle whose sides are  $a$ ,  $b'$  and  $r$  (or by the triangle whose sides are  $a'$ ,  $b$  and  $r$ ). The triangle  $ab'r$  is shown in Fig. 5, and it is called the *addition triangle*.

(b) **Resultant of any number of forces which act at a point.**  
**The addition polygon.**—The lines  $a$ ,  $b$ ,  $c$ , and  $d$  in Fig. 6 repre-

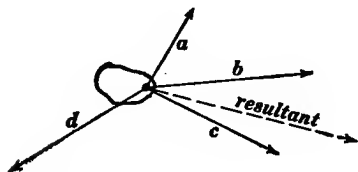


Fig. 6.

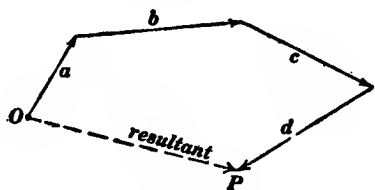


Fig. 7.

sent four forces acting on a body. To find the resultant or vector sum of these forces start from any point  $O$ , Fig. 7, and draw a line representing force  $a$ ; from the end of this line draw another line representing force  $b$ ; from the end of this line draw another line representing force  $c$ ; and so on, as indicated in Fig. 7. The point  $P$  is thus reached, and the line  $OP$  represents the vector sum or resultant of  $a$ ,  $b$ ,  $c$  and  $d$ .

If the resultant or vector sum of  $a$ ,  $b$ ,  $c$  and  $d$  is zero, the line  $OP$  is zero, or in other words the point  $P$  is at  $O$ .

**3. Resolution of a force into parts. Components of a force.**—The force  $F$  in Fig. 8 is the resultant of the two forces  $X$  and  $Y$ ; that is to say,  $X$  and  $Y$  are together exactly equivalent to  $F$ ; it is all the same whether the single force  $F$  or the two forces  $X$  and  $Y$  act on the body (as indicated in Fig. 9). Therefore  $X$  and  $Y$  are two parts into which the given force

$F$  may be divided. Now the parts of a thing are often called its *components*; thus the components of ordinary concrete are water, cement, sand and crushed stone. Similarly, the two forces

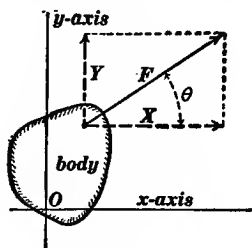


Fig. 8.

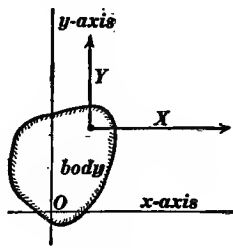


Fig. 9.

$X$  and  $Y$  are called the *components* of  $F$ ; in fact  $X$  and  $Y$  are called the *rectangular components* of  $F$  because  $X$  and  $Y$  are at right angles to each other. The rectangular components of a force are usually called, simply, the components of the force.

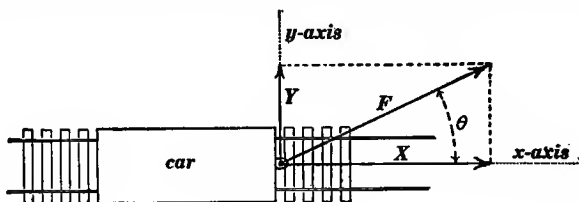


Fig. 10.

**Example.**—Figure 10 shows a force  $F$  acting on a car. The single force  $F$  is exactly equivalent to the two forces  $X$  and  $Y$ . The force  $Y$  has no effect\* in helping to move the car, and therefore the force  $X$  is the effective part of  $F$ .

**4. Addition of forces in terms of components.**—Consider the four forces  $a$ ,  $b$ ,  $c$  and  $d$  as indicated in Fig. 6, and let the components of these forces be as given in the accompanying table.

\* The force  $Y$  has an indirect influence on the motion of the car because it pulls the flanges of the car wheels against the rails and thus causes an increased amount of friction. This friction is a *new force* in addition to  $F$ ; but if  $F$  were the only force acting on the car its effectiveness in moving the car would be exactly the same as force  $X$ .

Force.	Components to Right.	Components Upwards.
<i>a</i>	+ 10 "pounds."	+ 11 "pounds."
<i>b</i>	+ 24 "	+ 4 "
<i>c</i>	+ 18 "	- 12 "
<i>d</i>	- 28 "	- 19 "
<i>r</i>	+ 24 "	- 16 "

Let *r* be the vector sum of *a*, *b*, *c* and *d*. Then the components of *r* are found by adding the respective components of *a*, *b*, *c* and *d* with due regard to algebraic sign, as indicated in the table.

**5. Action and reaction.**—When one pushes on an object by the hand one can feel a back-push on the hand by the object. Whenever one body *A* exerts a force on another body *B*, then the body *B* necessarily exerts an equal and opposite force on body *A*. Thus the upward pull of the rope on the hook at *a* in Fig. 1 is equal to the downward pull of the hook on the rope. *Action is equal to reaction and opposite in direction.*

Figure 11 represents a ladder leaning against a wall; *a* is the force exerted on the ladder by the wall (*a'* is the force exerted on the wall by the ladder), *b* is the gravity pull of the earth on the ladder (*b'* is the gravity pull of the ladder on the earth) and *c* is the force exerted on the ladder by the floor (*c'* is the force exerted on the floor by the ladder). A set of forces is always understood to be the forces which act on a given body, NOT THE FORCES EXERTED BY THE GIVEN BODY ON OTHER BODIES.

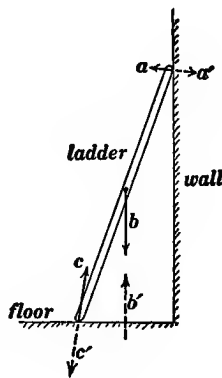


Fig. 11.

#### PROBLEMS.

The first step in the solution of a problem in force action is to make a diagram showing the body on which the forces act, and showing the forces themselves by arrows. A teacher may very properly throw into the waste basket any solution (?) of a prob-

lem in which no diagram is given or in which the diagram as given is not clearly intelligible.

1. A guy wire pulls on a vertical pole with a force of 200 "pounds." The angle between wire and pole is  $52^\circ$ . Find vertical and horizontal components of the 200-"pound" pull.

2. The vertical component of the pull of a guy wire is 400 "pounds," and the angle between the guy wire and the horizontal is  $50^\circ$ . Find the total pull of the guy wire.

3. Figure 12 shows a vertical strut which stands with its pointed lower end resting on a flat block of stone. Find the tension of the guy wire to give a tension of 250 "pounds" in the horizontal wire; and find compression of strut.

*Note.*—The pull of the horizontal wire balances and is therefore equal in value to the horizontal component of the pull of the guy wire. The upward push of the strut balances the downward component of the pull of the guy wire.

4. Two ropes placed symmetrically as shown in Fig. 13 support a weight of 1000 "pounds." Find tension in each rope.

*Note.*—Vertical component of pull of each rope is equal to one half of the supported weight.

5. Find tension of each rope in Fig. 13 if the angles be  $85^\circ$  instead of  $40^\circ$  as shown.

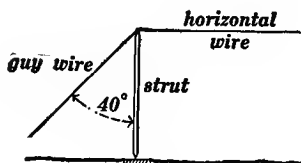


Fig. 12.

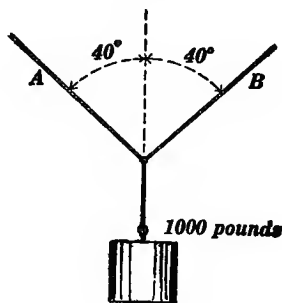


Fig. 13.

6. A horizontal wire and a guy wire pull on the top of a pole as shown in Fig. 14. The force  $A$  is 200 "pounds" and the force  $B$  is 300 "pounds." What is the resultant of  $A$  and  $B$  and what is its direction?

*Note.*—It is understood that this problem is to be solved by trigonometry, that is by solving the addition triangle, but for practical purposes it is as satisfactory to have the *two components* of the resultant as it is to have the *value* and the *direction* of the resultant. Therefore the problem may be solved by the addition of components as explained in Art. 4.

7. The following forces act on a post. (a) A northward force of 500 "pounds," (b) A force of 750 "pounds" towards the

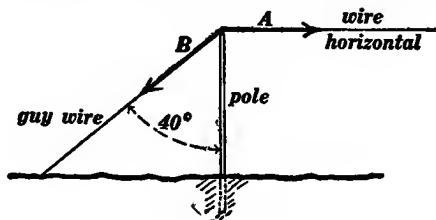


Fig. 14.

northeast, (c) A force of 200 "pounds" directed  $60^\circ$  south of east, and (d) A force of 150 "pounds" directed  $40^\circ$  west of south. Find northward and eastward components of the resultant.

8. Four forces, *a*, *b*, *c* and *d*, which act on a post have a resultant whose northward component is 1600 "pounds," and whose eastward component is 700 "pounds." Force *a* is 800 "pounds" towards northeast, force *b* is 900 "pounds" towards south, and force *c* is 2700 "pounds" towards southwest. What are the components of force *d*?

9. A stream flows southwards at a velocity of 2 miles per hour and a boatman rows towards the northwest at a velocity of 4 miles per hour. What is the actual velocity of the boat?

*Note.*—Velocities are added like forces, and the actual velocity of the boat is the vector sum of the southward velocity of 2 miles per hour and the northward velocity of 4 miles per hour.

10. A steamship is traveling at a velocity of 18 miles per hour in a direction  $42^\circ$  north of east. Find the northward and eastward components of the velocity.

---

6. **Balanced forces.**—Consider a set of forces which act at a point.\* The set of forces is *balanced* if the vector sum or resul-

\* The balance of forces which do not all act at a point is considered later.

tant of the set is zero. The study of balanced forces is called *statics*. The study of the behavior of a body when acted upon by unbalanced forces is called *dynamics*.

To balance each other a number of forces must, of course, all act on the same body. *Therefore equality of action and reaction has nothing whatever to do with the balance of forces.* Every trade has its *purchase aspect* and its *sales aspect*, and these two aspects are always equal. Thus if  $A$  buys \$10 worth of nails from  $B$ , then, of course,  $B$  sells \$10 worth of nails to  $A$ . To imagine that the forces which act on a **given body** must always balance each other because every action has an equal and opposite reaction is as absurd as to infer from the equal purchase and sales aspects of every trade that a **given merchant's** purchases cannot exceed his sales however much cash or credit he may have or however deficient he may be in good judgment. **When no force at all acts on a body, or when the forces which do act are balanced, the body either remains stationary or continues to move at uniform velocity in a straight line; or, conversely, if a body is either stationary or moving at uniform velocity along a straight line, the forces which act on the body are balanced.**

Thus the forward pull of a horse on a plow and the backward drag of the ground on the plow are equal when the plow is standing still or when the plow is moving forwards at constant velocity. While an elevator cage is moving upwards at constant velocity the upward pull of the elevator cable is equal to the combined downward pull due to gravity and friction.

**7. First condition of equilibrium.**—When a set of forces which act at a point are balanced the forces are related to each other in a certain way, and the statement of this relation is called the *first condition of equilibrium*.

I. If the forces  $a$ ,  $b$ ,  $c$  and  $d$  in Fig. 6 are balanced, their resultant  $OP$ , Fig. 7, is zero, or, in other words, the point  $P$  in Fig. 7 is at  $O$ . Therefore the balanced forces  $a$ ,  $b$ ,  $c$  and  $d$  are parallel and proportional to the sides of a closed polygon as indicated in Figs. 15 and 16.

II. Another statement of the relationship among the forces of a balanced set may be based upon Art. 4. The accompanying

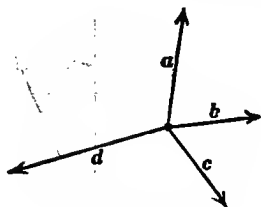


Fig. 15.

A balanced set of forces.

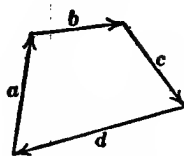


Fig. 16.

A closed polygon, sides parallel and proportional to lines in Fig. 15.

table exhibits the components of the forces which are shown in Fig. 15. The components of  $r$  (the resultant of  $a$ ,  $b$ ,  $c$  and  $d$ )

Force.	Components to Right.	Components Upwards.
$a$	+ 7 "pounds."	+ 10 "pounds."
$b$	+ 34 "	+ 6 "
$c$	+ 9 "	- 20 "
$d$	- 50 "	+ 4 "
$r$	0	0

are zero. The relation exhibited in the table may be most easily stated as follows:

$$\left\{ \begin{array}{l} \text{sum of all} \\ \text{components to right} \end{array} \right\} = \left\{ \begin{array}{l} \text{sum of all} \\ \text{components to left} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{sum of all} \\ \text{upward components} \end{array} \right\} = \left\{ \begin{array}{l} \text{sum of all} \\ \text{downward components} \end{array} \right\}$$

**Examples illustrating first condition of equilibrium.**—(a) A 100-pound ball is suspended by two ropes as indicated in Fig. 17. Find tension of rope  $A$  and tension of rope  $B$ .

The forces  $a$ ,  $b$  and  $c$  are parallel and proportional to the sides of the triangle shown in Fig. 18. Therefore since one side and two angles of this triangle are given the other sides  $a$  and  $b$  may be calculated.

It is usually simpler to solve a problem of this kind by the

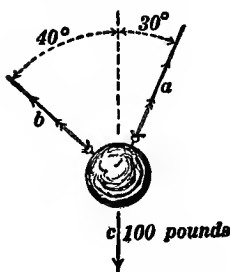


Fig. 17.

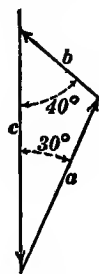


Fig. 18.

method of components as follows. Tabulating components, we get:

Forces.	Components to Right.	Upward Components.
<i>a</i>	$+ a \sin 30^\circ$ "pounds."	$+ a \cos 30^\circ$ "pounds."
<i>b</i>	$- b \sin 40^\circ$ "	$+ b \cos 40^\circ$ "
<i>c</i>	0 "	- 100 "

Therefore, according to Art. 7, we get:

$$a \sin 30^\circ = b \sin 40^\circ \quad (i)$$

$$a \cos 30^\circ + b \cos 40^\circ = 100 \quad (ii)$$

and the values of *a* and *b* are easily calculated from these equations.

(b) As a second example let us consider a 25-"pound" block which is being drawn steadily up a  $\theta^\circ$  incline by a force *A* as indicated in Fig. 19. The coefficient of sliding friction between block and plane is 0.2, and it is required to find the force *A*. The figure shows the force *A* exerted on the block by the rope and the force *W* exerted on the block by gravity; and the force exerted on the block by the plane is shown in two

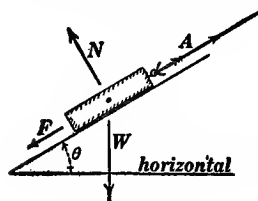


Fig. 19.

parts *N* and *F* where *N* is the force normal to the plane and *F* is the frictional drag which is parallel to the plane (and of

course opposite to the direction in which the block is moving). The ratio  $F/N$  is called the coefficient of sliding friction.

To solve this problem it is most convenient to consider components parallel to plane (up hill components) and components normal to plane (in direction of  $N$ ) and these components are as follows:

Forces.	Up-hill Components.	Components in Direction of $N$ .
$A$	+ $A$ "pounds."	0
$W$	- $W \sin \theta$ "	- $W \cos \theta$ "pounds."
$N$	0	$N$ "
$F$	- $F$ "	0

Therefore, according to Art. 7, we get

$$A = W \sin \theta + F \quad (i)$$

$$N = W \cos \theta \quad (ii)$$

and furthermore,  $\mu$  being the coefficient of friction;

$$\frac{F}{N} = \mu \quad (iii)$$

From these three equations the values of  $A$ ,  $N$  and  $F$  may be easily calculated if  $\theta$ ,  $W$  and  $\mu$  are known.

#### PROBLEMS.

**Note.** Throughout Art. 7 "components to right" and "components upwards" mean **components in any two directions at right angles to each other.** The terms *right* and *left*, and *upwards* and *downwards* are used only for brevity and clearness.

**11.** A 150-pound ball is hung by a rope, and a horizontal force  $F$  is exerted on the ball so that the rope makes an angle of  $30^\circ$  with the vertical. Find the pull of the rope on the ball and find  $F$ .

**Note.** Make a diagram in which the three forces *which act on the ball* are represented by arrows, and formulate the problem by considering *only* the forces which act on the ball.

**12a.** A 150-pound ball is hung by a rope and a horizontal force of 25 "pounds" acts on the ball. Find the pull of the rope and find the angle between the rope and the vertical.

*Note.* Make a diagram in which the three forces which act on the ball are represented by arrows, and formulate the problem by considering *only* the forces which act on the ball.

**12b.** A 200-pound cylinder rests in a trough of which one side is inclined  $45^\circ$  and the other side  $60^\circ$  above the horizontal. Assuming the sides of the trough to be frictionless, find the force exerted by the cylinder against each side.

**13a.** The three legs of a tripod make angles of  $30^\circ$  with the vertical. Find the compression in each leg when the tripod supports a 200-pound body.

*Note.* Consider only the forces which act on the head of the tripod and place the sum of all the upward forces equal to the sum of all the downward forces.

**13b.** Make a sketch showing a top view of the tripod of the

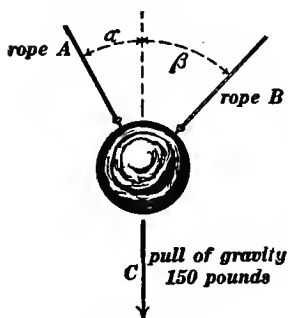


Fig. 20 (Old 21)

previous problem and draw three arrows representing the horizontal forces exerted by the three legs on the head of the tripod. How do you know that these three forces are balanced? What is the value of each?

**14.** A 150-pound ball is hung by two ropes as indicated in Fig. 20. Find pull of each rope when  $\alpha = 40^\circ$  and  $\beta = 55^\circ$ .

**15.** A 200-pound block rests on the rim of a wheel as shown in

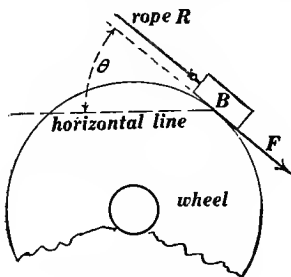


Fig. 21 (New)

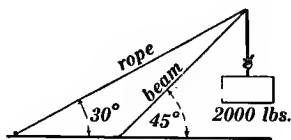


Fig. 22 (New)

Fig. 21. Find the force  $F$  (applied to the rim of the wheel) required to turn the wheel:—(a) When  $\theta = 0$ , (b) When  $\theta = 30^\circ$ , (c) When  $\theta = 45^\circ$ , and (d) When  $\theta = 60^\circ$ . Axle friction is to be ignored. Coefficient of friction between block and wheel 0.15.

16. Find the tension of the rope in each case in the previous problem.

17. The angle  $\theta$  in Fig. 19 is  $40^\circ$  and the coefficient of friction  $F/N$  is 0.2. Find the pull  $A$  which will allow the 200-pound block to slide steadily down the inclined plane.

18. A man without the usual tackle wishes to drag a boat up a sloping beach, and he ties a strong rope from the bow of the boat to a post which is 100 feet from the bow. Throwing himself sidewise against the middle of the rope he can exert a side force of 150 "pounds" on the rope and the middle of the rope is moved three feet sidewise. What is the pull of the rope on the boat?

19. Each link  $L$  and  $L'$  of a toggle is 10 inches long, and the distance  $QR$  in Fig. 23 is 19.8 inches. Find the components  $AA$

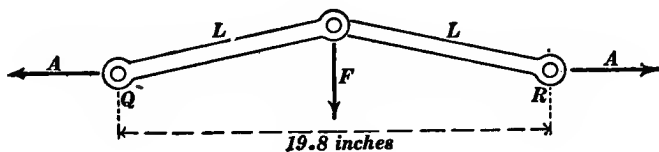


Fig. 23 (Old 23)

(parallel to  $QR$ ) of the forces exerted by the links of the toggle on the pins  $Q$  and  $R$  when the side force  $F$  is 50 "pounds," neglecting friction.

*Note.* To neglect friction is to assume that each link is in simple compression, pushing in its own direction at each end.

20. A span of telegraph wire weighs 15 "pounds," and the

wire at each pole is inclined  $5^\circ$  above the horizontal. Find the tension  $T$  of the wire near the poles and the tension  $H$  of the wire at the middle of the span.

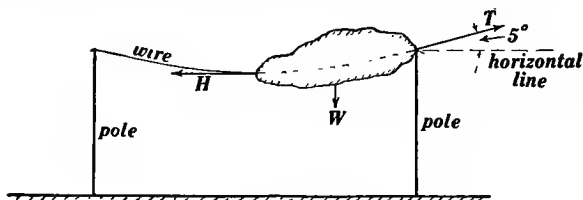


Fig. 24 (New)

*Note.* Look upon the wire in one-half of the span as a body  $B$  upon which a balanced set of forces  $T$ ,  $H$  and  $W$  act as indicated in Fig. 24.

21. The end  $Q$  of the beam in Fig. 25 has a roller bearing so that the push of the wall on the beam is at right angles to the wall (horizontal). The end  $Q$  of the beam is pushed downwards with a force of 100 "pounds." Find the compression of the beam.

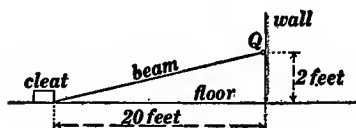


Fig. 25.

*Note.*—Consider three forces as acting on the roller, namely, (a) Push of wall, (b) Push of beam in the direction of the axis of the beam, and (c) Downward push of 100 "pounds."

8. Turning force or torque.—To set a top spinning by thumb and fore-finger *without throwing the top sidewise* it is necessary to

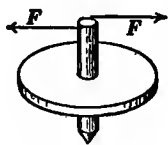


Fig. 26.

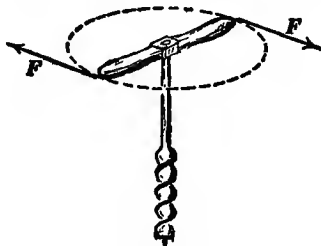


Fig. 27.

exert a pair of equal and opposite forces by thumb and fore-finger as indicated by the two arrows in Fig. 26. *A pair of equal and opposite forces thus exerted on a body constitutes a pure turning force or torque.* Such a pair of forces is sometimes called a *couple*. The two equal and opposite forces which are usually exerted on an auger handle as indicated in Fig. 27 constitute a *torque* or *couple*

**9. The principle of the lever. The balance of torque actions.**—The pairs of equal and opposite forces in Figs. 26 and 27 constitute what are called *pure torques*, and it is important to consider *how much* torque we have in any given case. This matter is most easily settled by considering the simple lever as follows: Two forces *A* and *B* act on the ends of a balanced lever as shown in Fig. 28, the weight of the lever itself being neglected for

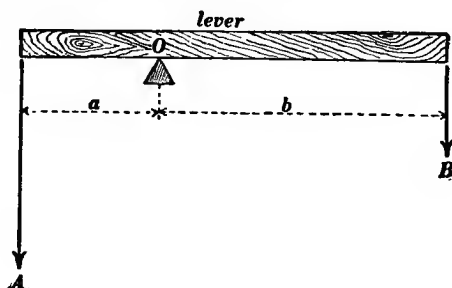


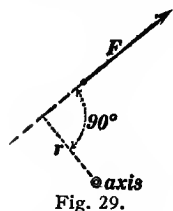
Fig. 28.

the sake of simplicity. The supporting fulcrum must of course exert a force on the lever to keep the lever from moving in the direction of the two forces *A* and *B*, *but the force exerted on the lever by the fulcrum has no tendency to turn the lever about O; therefore the torque action of A about O is evidently balanced by the torque action of B about O.*—Or, in other words, torque action of *A* about *O* must evidently be equal and opposite to torque action of *B* about *O*. Now the lever balances\* when

$$Aa = Bb \quad (i)$$

\* Weight of lever is assumed to be negligible in this statement.

where  $a$  and  $b$  are the distances shown in Fig. 28. This equation is easily verified by experiment, and, in accordance with this equation, we may take the products  $Aa$  and  $Bb$  as expressions for the respective torque actions of the forces  $A$  and  $B$  about  $O$ , force  $A$  being at right angles to  $a$ , and force  $B$  being at right angles to  $b$ . Therefore, in general, we have:



*The torque action of a given force about any chosen axis is equal to the product  $Fr$ , where  $F$  is the value of the force and  $r$  is what is called the "lever arm" of the force as indicated in Fig. 29.—*

The product  $Fr$  is sometimes called the *moment of the force* about the chosen axis.

**10. Second condition of equilibrium.**—Consider a set of forces acting on a body. The torque action of each force *about a chosen axis* may be calculated as above explained, and, if the set of forces has no tendency to set the body rotating about the chosen axis we must have:

$$\left\{ \begin{array}{c} \text{sum of clockwise} \\ \text{torques} \end{array} \right\} = \left\{ \begin{array}{c} \text{sum of counter-clockwise} \\ \text{torques} \end{array} \right\}$$

This condition is called the *second condition of equilibrium*.

Let us consider a set of forces which *do not* all act at a point, then this set of forces is *completely balanced* if the **first condition of equilibrium** is satisfied (see Art. 7), and if the above condition is satisfied with respect to any axis whatever.

### 11. Examples illustrating the second condition of equilibrium.

(a) Figure 30 represents a beam or lever pivoted at the point  $O$ . The total gravity pull on the beam itself is  $L$  "pounds" (weight of beam) and this may be thought of as a single force acting at the point  $G$  which is called the *center of gravity* of the beam. Other vertical forces  $A, B, D, E$  and  $F$  act on the beam as indicated, and the torque-actions about  $O$ , being tabulated, are as follows, forces being in "pounds" and lever arms  $a, b$ , etc., being in feet:

Force.	Clockwise Torques.	Counter-clockwise Torques.
<i>A</i>	_____	<i>Aa</i> "pound"-feet.
<i>B</i>	_____	<i>Bb</i> "
<i>L</i>	<i>Lg</i> "pound"-feet.	_____
<i>D</i>	<i>Dd</i> "	_____
<i>E</i>	<i>Ee</i> "	_____
<i>F</i>	_____	<i>Ff</i> "

and if the beam balances on the pivot at *O* we must have:

$$Lg + Dd + Ee = Aa + Bb + Ff$$

(b) Figure 31 represents a trunk which is being turned over

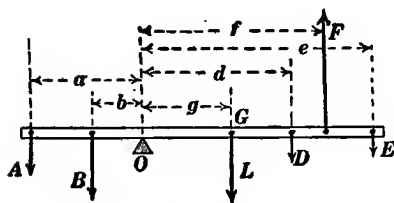


Fig. 30.

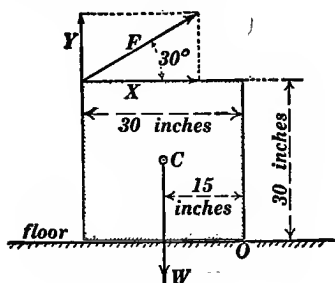


Fig. 31.

by a force *F*. It is required to find the value of *F*, the weight *W* of the trunk being 150 "pounds."

The two components of *F* are  $X = F \cos 30^\circ$  and  $Y = F \sin 30^\circ$ , and the torque actions about *O* of *W*, *X* and *Y* are:

Forces.	Clockwise Torques.	Counter-clockwise Torques.
<i>X</i>	$30 X$ "pound"-inches.	_____
<i>Y</i>	$30 Y$ "	_____
<i>W</i>	_____	$150 \times 15$ "pound"-inches

Therefore we must have:

$$30(X + Y) = 2250$$

or, using values of *X* and *Y* in terms of *F*, we have:

$$1.366 \times 30 \times F = 2250$$

from which the value of *F* may be found

(c) Figure 32 shows a simple wall bracket resting on the floor at  $O$ , fixed to the wall by a single screw which exerts a horizontal pull  $F$  on the bracket, and supporting a 500-“pound” weight. Find the force  $F$ . If the screw should give way the bracket would turn about the point  $O$ , and the torque action of  $F$  about  $O$  must be equal and opposite to the torque action of the 500-“pound” force about  $O$ .

## PROBLEMS.

22. Find the value of  $F$  in Fig. 30 to balance the lever,  $A = 60$  “pounds,”  $B = 120$  “pounds,” weight of beam “30 pounds,”  $D = 25$  “pounds,”  $E = 100$  “pounds,”  $a = 4$  feet,  $b = 2$  feet,  $g = 1\frac{1}{2}$  feet,  $d = 5$  feet,  $e = 7$  feet, and  $f = 6$  feet.

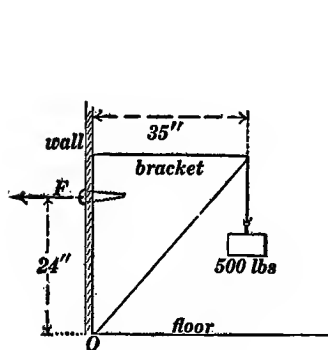


Fig. 32.

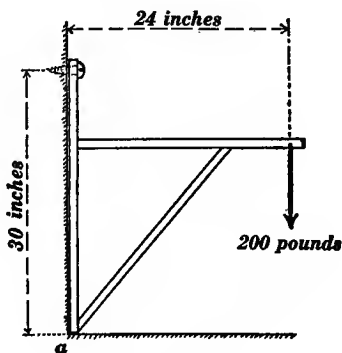


Fig. 33.

23. Find the horizontal pull of the screw on the bracket in Fig. 33, neglecting weight of bracket itself.

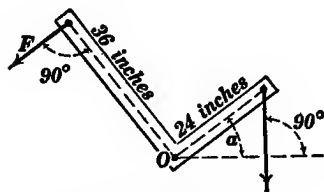
24. Find the force  $F$  required to balance the bell-crank lever about the pivot  $O$  in Fig. 34, the angle  $a$  being  $25^\circ$ .

25. A uniform board 16 feet long and weighing 25 “pounds” is held in a horizontal position with one end resting on a table, and a point 6 feet from the other end resting on the hand. Find the upward force exerted by the hand.

*Note.*—Apply the second condition of equilibrium, and reckon torques about

the point of application of the force whose value you do not know and do not wish to find.

26. The oarsman shown in Fig. 35 bends his legs so as to reduce to 24 inches the distance which is shown as 36 inches in the



The vertical force acting on the short lever arm is 200 "pounds."

Fig. 34.

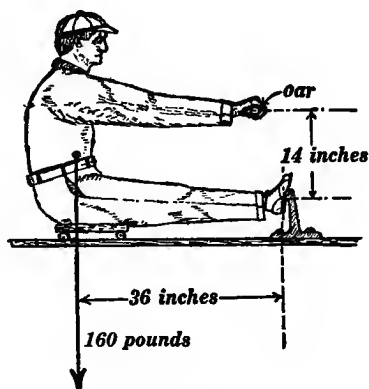


Fig. 35.

figure. Find the greatest horizontal force he can exert on the oar without lifting himself off the seat.

*Note.*—Of course the force to be used in formulating this problem is the force with which the oar pulls on the man. An excessive pull lifts the man off his seat, and his body turns about the point (axis) where his feet rest against the cleat.

27. A draft horse weighs 2,200 pounds and the center of gravity of the horse is, let us say,  $3\frac{1}{2}$  feet forward of the point where the horse's hind hoofs rest on the ground. The traces or tugs are horizontal and 20 inches above the ground. What is the maximum pull that can be exerted by the horse if his hind feet do not slip?

28. An arrangement of two levers is shown in Fig. 36. Find the force at  $W$  required to balance the 2-pound weight.

*Note.*—In considering the equilibrium of beam  $A$  it is necessary to think of the force with which  $B$  pulls down on  $A$ . This force being found, we know the force with which  $A$  pulls up on  $B$ , and this force must be thought of in considering the equilibrium of  $B$ .

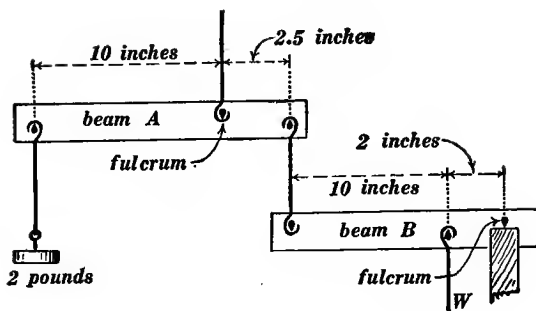


Fig. 36.

29. Figure 37 shows a common form of geared windlass. The radius of the crank is 2 feet, and the radius of the drum upon which the rope winds is 6 inches. A force of 100 "pounds" is applied at the crank handle in a direction at right angles to the crank radius. Find the weight  $A$  required to produce equilibrium, ignoring friction.

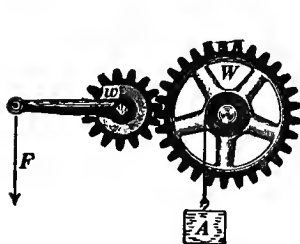


Fig. 37

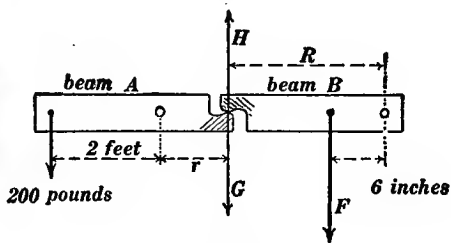


Fig. 38.

*Note.*—The arrangement in Fig. 37 is exactly equivalent to the arrangement in Fig. 38; the ratio  $\frac{r}{R}$  in Fig. 38 being equal to the ratio  $\frac{n}{N}$  in Fig. 37, where  $n$  is the number of cogs on the small gear wheel  $w$  and  $N$  is the number of cogs on the large gear wheel  $W$ .

The distances  $r$  and  $R$ , Fig. 38, are the radii of what are called the *pitch circles* of the two gear wheels.

The arrows  $G$  and  $H$  represent the forces with which the beams act and react on each other.

**12. Examples illustrating application of both conditions of equilibrium.**—(a) Figure 39 shows a traveling crane running on

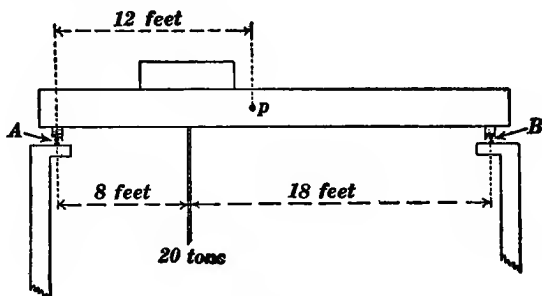


Fig. 39.

rails at *A* and *B*, and carrying a weight of 20 "tons" on its hoisting rope as indicated. The crane itself weighs 6 "tons" and its center of gravity is at the point *p*. Let *A* be the upward force exerted on the crane by rail *A*, and let *B* be the upward force exerted on the crane by rail *B*. Find the values of *A* and *B*.

Placing sum of all upward forces equal to sum of all downward forces we get:

$$A + B = 26 \text{ "tons"} \quad (i)$$

To apply the second condition we may reckon torque actions about any axis whatever. Let us choose rail *A* for this axis. Then tabulating clockwise and counter-clockwise torques about rail *A* and placing sum of former equal to sum of latter we get:

$$(20 \text{ "tons"} \times 8 \text{ feet}) + (6 \text{ "tons"} \times 12 \text{ feet}) = B \times 26 \text{ feet} \quad (ii)$$

The value of *B* may be found from (ii), and then the value of *A* may be found from (i).

(b) Figure 40 represents a 100-"pound" bench which is being steadily dragged along a floor by a horizontal pull *P*. The coefficient of friction between each bench leg and the floor is 0.2. Find upward push of floor on each bench leg (forces *R* and *L*) and find value of *P*.

The normal forces exerted by the floor against the sliding surfaces (ends of bench legs) are  $R$  and  $L$ , and the horizontal forces (parallel to direction of sliding surfaces) which oppose the sliding are  $aR$  and  $aL$ , where  $a$  is the coefficient of sliding friction.

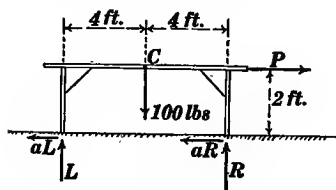


Fig. 40.

Placing sum of all forces to right equal to sum of all forces to left, we get:

$$P = aR + aL \quad (i)$$

Placing sum of all upward forces equal to sum of all downward forces, we get:

$$R + L = 100 \text{ "pounds"} \quad (ii)$$

Choosing point C, let us say, as axis about which torques are to be reckoned, tabulating clockwise and counter-clockwise torques, and placing sum of clockwise torques equal to sum of counter-clockwise torques, we get:

$$4L + 2aL + 2aR = 4R \quad (iii)$$

and the three equations (i), (ii) and (iii) permit the easy calculation of values of  $R$ ,  $L$  and  $P$ .

**Remark.**—We may apply the second condition of equilibrium *twice* about different axes, and not use the first condition of equilibrium at all in example (a). We may apply second condition of equilibrium *three times* about different axes, and not use first condition of equilibrium at all in example (b). One is not entirely free in one's choice of the *two* or *three* different axes. Let the reader think this out for himself.

#### PROBLEMS.

30. Figure 41 represents a 14-foot beam carrying a load of 500 "pounds." The beam itself weighs 75 "pounds" and its

center of gravity is at its middle point. Find the forces  $A$  and  $B$ .

31. Find the force exerted on the trunk by the floor at the point  $O$  in Fig. 31.

32. A 16-foot ladder

rests against a wall with its base 4 feet from the wall. The weight of the ladder is 100 "pounds," and the center of gravity of the ladder is 7 feet from its lower end. Assuming the force  $a$ , Fig. 11, which is exerted on the ladder by the wall to be horizontal, find the value of  $a$  and find the components of the force exerted on the ladder by the floor.

*Note.*—The problem of three balanced forces may be in some cases greatly simplified by making use of the proposition that *the lines of action of three balanced forces must always intersect at a point*, except when the forces are parallel.

33. A horizontal force  $h$  is exerted on a swinging crane where the upper pivot end of the crane passes through the ceiling, the dimensions of the crane are as shown in Fig. 42, and the load on the crane is 6 "tons" as shown. Find the value of  $h$ , and find vertical and horizontal forces exerted on the lower pivot end of the crane by the floor.

34. Find the tension of the rope  $A$  and find the value and direction of the force exerted on the wall at  $D$  in Fig. 43.

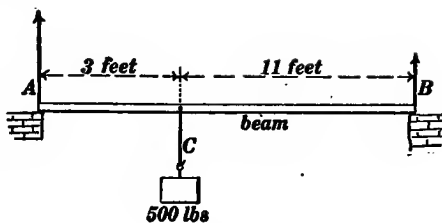


Fig. 41.

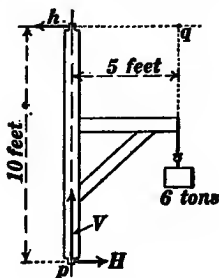


Fig. 42.

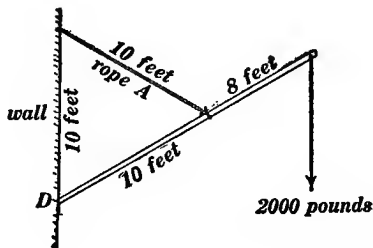


Fig. 43.

35. Find the force  $F$  required to push a 50-"pound" carriage up a slope as shown in Fig. 44, and find the values of the two

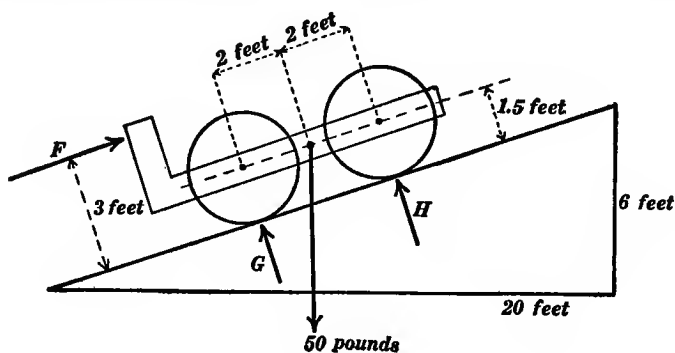


Fig. 44.

forces  $G$  and  $H$ . Ignoring friction at axles, the forces  $G$  and  $H$  are to be considered as at right angles to the ground.

36. The steam in a steam engine cylinder pushes with a force of 250,000 pounds on the piston. The positions and lengths of connecting rod and crank radius are shown in Fig. 45. Find

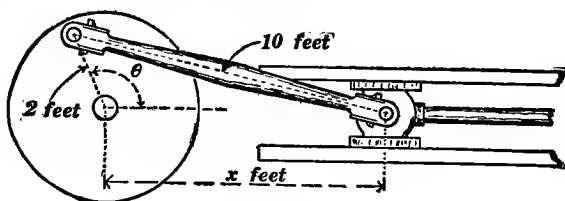


Fig. 45.

the force with which the cross-head pushes sidewise against the cross-head guides, find the compression in the connecting rod, and find the torque exerted on the crank-shaft when the angle  $\theta$  is  $120^\circ$ , ignoring friction.

37. The elevator cage which is shown in Fig. 46 (weight 3000 "pounds," center of gravity at  $C$ ) is moving upwards at constant velocity, and the coefficient of sliding friction  $a$  be-

tween cage and each side guide is 0.1. Find tension of cable, and find horizontal forces  $R$  and  $L$ .

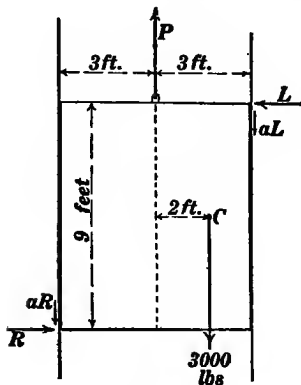


Fig. 46.

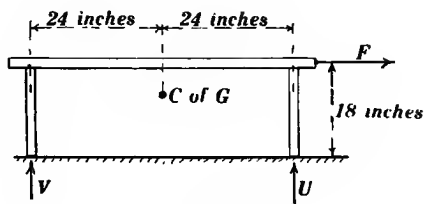


Fig. 47.

38. Find tension of cable and horizontal forces  $L$  and  $R$  when the elevator cage of Fig. 46 is moving downwards at constant speed.

39. Find the forces  $F$ ,  $U$  and  $V$  in Fig. 47. The bench weighs 200 "pounds," it is moving at constant speed and the coefficient of friction between bench legs and floor is 0.2.

**13. Complete resultant of a set of forces which do not all act at the same point.**—We are here again concerned with unbalanced sets of forces. Consider such a set of forces. If all of the forces act at the same point, the set is equivalent to a single force acting at that point, and this single force is called the resultant of the set as explained in Art. 2.

If all the forces of a set do not act at the same point, then the set *may* or *may not* be equivalent to a single force (*may* or *may not* have a resultant); if the set constitutes a pure torque or couple it has no resultant, if the set does not constitute a pure torque or couple it does have a resultant.

The single force which is equivalent to a set of forces which

do not all act at the same point is usually spoken of as the *complete resultant* of the set because we must know not only its magnitude and direction but also where it acts (the line along which it acts).

The *value* and *direction* of the complete resultant are given by the value and direction of the vector sum of the forces, to be found as explained in Art. 2 or in Art. 4.

The *line of action* of the complete resultant is determined by

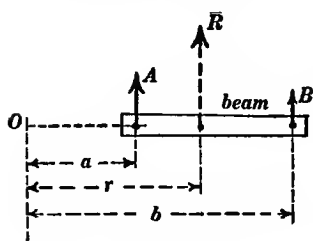


Fig. 48.

the condition that the torque action of the complete resultant about any chosen axis must be equal to the combined torque action about that axis of all the forces of the set.

#### 14. Examples illustrating the determination of complete resultants.—

(a) Figure 48 shows two parallel forces *A* and *B* acting on a beam, and it is desired to find the complete resultant *R*.

The value of *R* is

$$R = A + B$$

Let us choose any point (axis) *O* about which to reckon torque actions. The combined torque action of *A* and *B* about *O* is  $Aa + Bb$ , and the torque action of *R* about *O* is  $Rr$ . Therefore we must have

$$Rr = Aa + Bb$$

from which the value of *r* may be calculated, *R*, *A*, *B*, *a*, and *b* being known.

(b) Four forces *A*, *B*, *C* and *D* act on a table top as indicated in Fig. 49 (which is a top view). Find the single force *R* which is equivalent to the set and find its point of application.

Using the method of Art. 4 we find the components of *R* to be:

$$x\text{-component of } R = X = 21 \text{ "pounds"}$$

$$y\text{-component of } R = Y = 16 \text{ "pounds."}$$

Let  $x$  and  $y$  be the coördinates of any point  $p$  in the line of action of  $R$ , then  $R$  may be thought of as acting at  $p$  as indicated in Fig. 50. Let us reckon torque actions about the

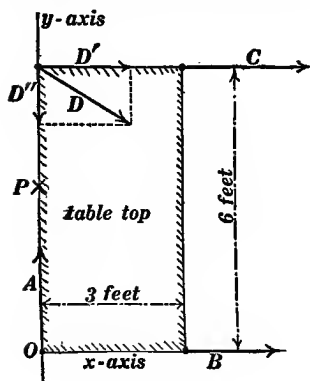


Fig. 49.

$A = 23$  lbs.  
 $B = 10$  lbs.  
 $C = 7$  lbs.  
 $D' = 4$  lbs.  
 $D'' = 7$  lbs.

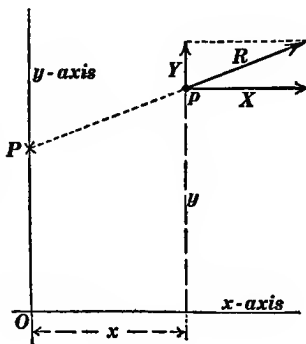


Fig. 50.

point  $O$ . The torque action of  $R$  (considering clockwise torques as positive) is  $Xy - Yx$ , and the combined torque action of  $A$ ,  $B$ ,  $C$  and  $D$  is  $+66$  "pound"-feet. Therefore  $Xy - Yx = 66$  "pound"-feet, or, using  $X = 21$  "pounds" and  $Y = 16$  "pounds," we get

$$21y - 16x = 66 \text{ feet}$$

which is the equation to the line  $Pp$ , Fig. 50, and the distance  $OP$  is  $66 \text{ feet}/21$  or  $3.14$  feet.

If we choose we may think of the force  $R$  as acting at the point  $P$  in Fig. 50, and accordingly  $R$  is a force whose  $x$ -component is  $21$  "pounds," whose  $y$ -component is  $16$  "pounds" and whose point of application is the point  $P$  in Fig. 50 or the point  $P$  in Fig. 49 which is  $3.14$  feet from  $O$ .

#### PROBLEMS.

40. Find the complete resultant of the two forces in Fig. 51. That is, find the value and direction of the single force which is entirely equivalent to the two given forces, and find the distance from  $O$  to its point of application.

41. Find the complete resultant of the two forces in Fig. 52.  
 42. A force  $A$  of 20 "pounds" pushes sidewise on one end of

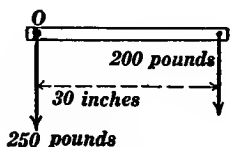


Fig. 51.

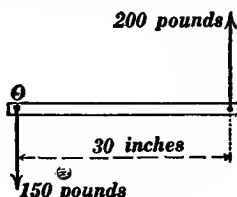


Fig. 52.

a table as shown in Fig. 53. Find the direction, value and point of application of a second force  $B$  such that the two forces together may be equivalent to a force  $R$  of five "pounds" oppo-

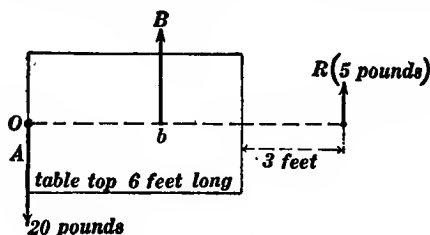


Fig. 53.

site in direction to  $A$  and acting (on an extension) at a point three feet beyond the end of the table.



### WORK.

#### 15. Active forces and inactive forces. Definition of Work.—

Nothing is more completely established by experience than the necessity of employing an active agent such as a horse or a steam engine or a water wheel to drive the machinery of a mill or factory, to draw a car or to propel a boat, and the common feature of every case in which motion is maintained is that *a force is exerted on a moving body and in the direction in which the body moves*. Such a force is called an **ACTIVE FORCE**.

An active force can be traced in every case to a substance which grows shorter under tension or to something which grows larger under compression. Thus a muscle under tension grows shorter, the steam in a steam engine cylinder grows larger or expands under pressure, and so on.

According to the above definition the force with which a boy pulls on a sled is called an active force, but the boy does not move with respect to the sled, and so far as the relation between boy and the sled is concerned, they might as well be aboard a train and both moving along together, or the boy might as well be pulling forwards on the rear door knob of the car in which he is riding. The real activity is in the muscles of the boy's legs as, of course, everyone knows.

A force which acts on a stationary body may be kept up indefinitely without effort or cost. Thus a weight suspended by a string continues indefinitely to pull on the string, the main spring of a watch will continue indefinitely to exert a force on the wheels of the watch if the watch is stopped. An **INACTIVE FORCE** is a force which acts on a body which is stationary or on a body which moves in a direction at right angles to the force. Thus the downward push of a driver on the seat of a wagon which travels along a level road is an inactive force, the forces with which the spokes of a rotating wheel pull inwards on the rim of the wheel are inactive forces.

**An active force is said to do work**, and the amount of work done is equal to  $Fd$ , where  $d$  is the distance that the body has traveled in the direction of the force  $F$  which acts on the body. If the movement  $d$  is not in the direction of  $F$  then the amount of work done is  $Fd \cos \theta$ , where  $\theta$  is the angle between  $F$  and  $d$ .

When a torque  $T$  acts on a body (on a wheel, for example) an amount of work  $T\phi$  is done when the wheel turns through an angle of  $\phi$  radians, axis of rotation being coincident with axis of torque.

**Units of work.**—*The unit of work is the work done by unit force while the body on which the force acts moves unit distance in the direction of the force.*

The *dyne-centimeter* is the *c.g.s.* unit of work and it is the work done by a force of one dyne while the body upon which the force acts moves one centimeter in the direction of the force. The dyne-centimeter of work is called the *erg*. The erg is for

most purposes an inconveniently small unit of work; therefore a multiple of the erg is extensively used, namely, the *joule*, which is equal to ten million ergs ( $10^7$  ergs).

The *foot-“pound.”* The “pound” is the pull of gravity on a one-pound body in London, and the work done by this force while the body upon which it acts moves one foot in its direction is called a *foot-“pound.”* The foot-“pound” is extensively used as the unit of work among English-speaking engineers.

**16. Rate of doing work. Definition of Power.**—The rate at which an agent does work is called the *power* of the agent. Thus a horse exerts a pull of 100 “pounds” on a plow and the plow moves 180 feet in the direction of the force in 60 seconds. The work done is 180 feet  $\times$  100 “pounds” which is 18,000 foot-“pounds,” and, dividing the work done by the time required for doing it, we get 300 foot-“pounds” per second as the rate at which the horse does work.

**Units of power.**—Power may of course be expressed in ergs per second or in foot-“pounds” per second, as in the above example. The units of power which are most extensively used, however, are the *watt* and the *horse-power*.

The *watt* is defined as one joule per second (ten million ergs per second). The kilowatt is 1,000 watts.

The *horse-power* is defined as 746 watts, or very nearly 550 foot-“pounds” per second.

**Power developed by an active force.**—Consider a force  $F$  acting on a body which moves in the direction of the force at velocity  $v$ . During  $t$  seconds the body moves through the distance  $vt$ , and the amount of work done is  $F \times vt$ . Therefore, dividing the amount of work done,  $Fvt$ , by the time  $t$  during which it is done, we have the rate  $P$  at which the work is done, that is,  $P = Fv$ . If  $F$  is expressed in dynes and  $v$  in centimeters per second, then  $P$  is expressed in ergs per second. If  $F$  is expressed in “pounds” and  $v$  in feet per second, then  $P$  is expressed in foot-“pounds” per second.

**Example.**—A passenger locomotive exerts a pull of 6,000 “pounds” on a train, the velocity of the train is 90 feet per second, and the net power developed by the locomotive (not counting the power required to propel the locomotive itself) is 540,000 foot-“pounds” per second, or 991 horse-power.

When a torque  $T$  acts on a body (on a wheel, for example) an amount of power equal to  $Ts$  is developed when the wheel rotates at a spin velocity of  $s$  radians per second, if the axis of spin coincides with the axis of the torque.

**17. Power-time units of work.**—Most practical measurements relating to work are measurements of power, and it has therefore come about that a given amount of work done is often expressed as the product of power and time.

The *watt-hour* is the amount of work done in one hour by an agent which does work at the rate of one watt.

The *kilowatt-hour* is the amount of work done in one hour by an agent which does work at the rate of one kilowatt.

The *horse-power-hour* is the amount of work done in one hour by a agent which does work at the rate of one horse-power.

#### ENERGY.

**18. Definition of energy. Kinetic energy\* and potential energy.**—Any agent which is able to do work is said to possess *energy*, and the amount of energy an agent has in store is equal† to the amount of work the agent can do. Suppose that a post, standing beside a railway track, is to be pulled out of the ground; can a car-load of stone be made to do the work? Certainly it can. All that is necessary is to have the car moving past the post and to throw over the post a loop of cable which is attached to the moving car. A moving car is able to do work; and when it does work its velocity is reduced, and its store of energy decreased. The energy which a body stores by virtue of its velocity is called the *kinetic energy* of the body.

\* A discussion of kinetic energy belongs properly to the subject of dynamics. See Art. 37, Chapter II.

† When we come to consider heat as a form of energy this definition will have to be modified because the heat energy a body stores cannot all be converted into mechanical work.

It is also a familiar fact that a weight can drive a clock, but in doing so the position of the weight changes and its store of energy is reduced. The energy which a body stores by virtue of its position is called the *potential energy* of the body.

The physical reality which lies behind the terms kinetic energy and potential energy can perhaps be shown most clearly by considering a bicycle rider. Suppose that the rider faces a steep hill or a sandy stretch of road where he is called upon to do an unusual amount of work. Every bicycle rider realizes the advantage of having a large velocity in such an emergency. This *advantage of velocity* is called kinetic energy. Or suppose that a bicycle rider wishes to use his whole strength, or more if he had it, in covering a certain distance. Every bicycle rider realizes the advantage of being on top of a hill in such an emergency. This *advantage of position* is called potential energy.

**19. The principle of the conservation of energy.**—Work is done when a weight is lifted, and the work is returned when the weight comes back to its initial position. Work is done when a car is set in motion, and the work is returned when the car is stopped. Work is done when a spring is bent, and the work is returned when the spring unbends. Work is done when air is compressed in a bicycle pump, and the work is returned if the piston is raised and the gas allowed to expand. *In all these examples friction\* is assumed to be non-existent so that the action in each example may be reversible as stated.* Friction, however, is *irreversible*; when a box is dragged along the floor work is done, but, as everyone knows, the box does not tend to fly back to its initial position and return the work which has been done upon it!

(a) When irreversible effects, like friction, do not exist, the work which is done **ON**† a body is completely recoverable as work done **BY** the body, or, in other words, no energy is lost.

\* Friction and all other irreversible effects. Thus the air which is compressed in a bicycle pump is heated, and we assume that this heated air is not cooled by contact with the cold metal walls of the pump cylinder.

† Some of the work done on the body may be converted into Heat, as in the compression of a gas, but Heat which is produced by reversible action is always completely recoverable as mechanical work.

(b) When irreversible effects, like friction, do exist, then a portion of the work done **ON** a body is left unrecovered in the form of Heat when the body is brought back to its initial condition. This Heat is a form of energy and **no energy has been destroyed** *although the availability of the energy for the doing of mechanical work has been very greatly reduced.* Irreversible effects, like friction, do not destroy energy, but they always “degrade” energy (reduce the availability of the energy for the doing of mechanical work).

(c) In every case, that is, in the ideal case *a* and in the real case *b*, energy is conserved, that is to say, the total amount of energy in existence is never altered. This fact is called *the principle of the conservation of energy.*

#### PROBLEMS.

43. A 165-pound man climbs a height of 40 feet in 11 seconds. How much work is done, and at what rate? Express the work in foot-“pounds,” and in joules; and express the power in foot-“pounds” per second, in horse-power, and in watts.

44. A horse pulls upon a plow with a force of 100 “pounds” and travels 3 miles per hour. What power is developed? Express the result in foot-“pounds” per second, in horse-power, and in watts.

45. The engines of a steamship develop 20,000 horse-power, of which 30 per cent. is represented in the forward thrust of the screw in propelling the ship at a speed of 17 miles per hour. What is the forward thrust of the screw in “pounds”?

*Note.*—The useful part of the power developed by the engines of a steamship is represented by the forward thrust of the propeller shaft against the framework of the ship, and the useful power is equal to the product of this force times the velocity of the ship

46. An electric motor has an efficiency of 80 per cent. and electrical energy costs 5 cents per kilowatt-hour. How much does the output of the motor cost per horse-power-hour, ignoring interest on cost of motor and its depreciation?

*Note.*—The efficiency of a motor is its output of power divided by its intake of power.

47. A 1,000 horse-power boiler and engine plant costs about \$70,000 complete, including land, building, boilers, engines and auxiliary apparatus such as pumps and feed water heaters. The cost of operating this plant continuously, night and day, is as follows:

Interest on investment . . . . .	5 per cent per annum.
Depreciation . . . . .	10 " " " "
Maintenance and repairs . . . . .	4 " " " "
Taxes and insurance . . . . .	2 " " " "
Labor \$30 per day, 365 days in year.	
Coal \$2.00 per ton.	

The average demand for power is 50 per cent. of the rated power output of the plant, that is 500 horse-power, and the consumption of coal is  $2\frac{1}{2}$  pounds per horse-power-hour. Find the cost of each horse-power-hour delivered by the engine.

48. The above engine will drive a 700 kilowatt dynamo, that is a dynamo capable of delivering 700 kilowatts. The cost of dynamo, station wiring and switch-board apparatus is \$20,000. The average output of the dynamo is 350 kilowatts (corresponding to 500 horse-power output of engine). Calculate the cost of electrical energy per kilowatt-hour at the station, allowing 21 per cent. for interest, depreciation, etc., on the electrical machinery and allowing \$5 per day for extra labor.

49. Prove that a torque  $T$  does an amount of work  $T\theta$  when the body (a wheel, for example) on which  $T$  acts turns  $\theta$  radians about the axis of  $T$ .

50. A shaft which rotates at a speed of 300 revolutions per minute transmits 100 horse-power. What amount of torque acts to twist the shaft?

51. To determine the power developed by an electric motor a brake is applied to the motor pulley, and the observed brake pull on the periphery of the pulley is 165 "pounds." The diameter of the pulley is 16 inches and its speed under test is 900 revolutions per minute. What amount of power is developed by the motor?

## APPLICATION OF THE PRINCIPLE OF WORK TO THE SOLUTION OF PROBLEMS IN STATICS.

**20. The principle of virtual work.**—Consider a body upon which a set of balanced forces act, and imagine the body to move in any way whatever. Some of the forces will help the movement and do work in a *positive sense* (positive work being work done **ON** the body), some of the forces will oppose the movement and do work in a *negative sense* (negative work being work done **BY** the body), and **the total work done by all the forces will be zero**. The movement must be assumed not to introduce or create new forces in addition to the forces which are being studied (the forces which belong to the given set), a condition which is usually expressed by speaking of the movement as a *virtual movement*, meaning a movement which is merely thought of but which does not actually take place.

**Example 1.**—If the lever in Fig. 28 were to be turned clockwise about the fulcrum  $O$  through a very small angle  $\theta$ , the end  $A$  of the lever would move up a distance  $a\theta$  and an amount of work  $Aa\theta$  would be done **BY** the lever, and the end  $B$  would move down a distance  $b\theta$  and an amount of work  $Bb\theta$  would be done **ON** the lever. Then, according to the principle of virtual work, we would have  $Aa\theta = Bb\theta$ , or  $Aa = Bb$ .

**Example 2.**—A wrench is applied to a nut on a bolt, and a force  $F$  is applied to the wrench as indicated in Fig. 54. Find the total force  $N$  which will give a balance, friction being ignored. If the nut were to be given one complete turn, the point  $P$  would move a distance  $2\pi r$  in the direction of  $F$  and an amount of work  $2\pi r F$  would be done **ON** wrench and nut.

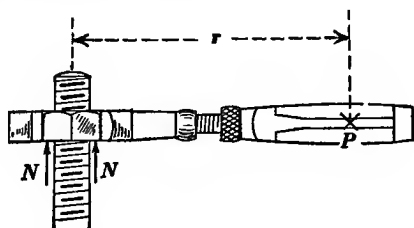


Fig. 54.

Force  $F$  is applied at point  $P$  and it acts towards the reader.

The nut would travel a distance  $p$  (called the *pitch* of the screw) and an amount of work  $pN$  would be done **BY** wrench and nut. Then, according to the principle of virtual work, we would have  $2\pi rF = pN$ , or  $N = 2\pi rF/p$ .

## PROBLEMS.

52. A 250-“pound” barrel is drawn up an incline by pulling on a rope at  $a$  as indicated in Fig. 55. What amount of work is done in lifting the barrel 3 feet, ignoring friction? How far does the end  $a$  of the rope move while the barrel travels up the plane (10 feet)? How much force must be exerted on the rope at  $a$ , ignoring friction?

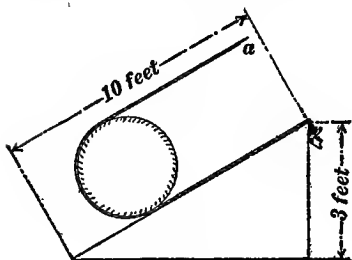


Fig. 55.

53. A screw jack lifting a weight of 10,000 “pounds” is turned by a capstan rod, and the force required to turn the screw is 75 “pounds,” lever arm being 18 inches. The pitch of the

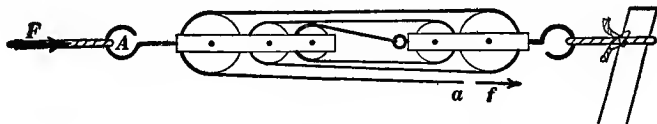


Fig. 56.

screw is  $\frac{1}{4}$  inch. What fraction of the work spent in turning the screw is actually utilized in lifting the weight, and what fraction is lost in friction?

54. Figure 56 is a diagram showing a block and tackle. Find movement of  $A$  due to one foot of movement of  $a$ , and calculate the force  $F$  due to  $f = 1000$  “pounds,” ignoring friction.

55. A large water tank has 120 staves

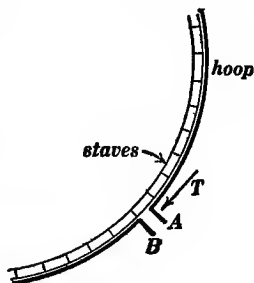


Fig. 57.

each 6 inches wide and a steel hoop has tension  $T = 15,000$  "pounds." Calculate inward push of hoop on each stave.

*Note.*—Imagine  $B$ , Fig. 57, to be stationary. Then if  $A$  were to be moved 0.01 foot towards  $B$  calculate work done ON hoop by force  $T$ . Then calculate inward movement of each stave when circumference of hoop (or tank) is shortened by 0.01 foot. Then calculate force which must push inwards on each stave, using principle of virtual work.

## CHAPTER II.

### DYNAMICS OF TRANSLATORY MOTION.

**21. Translatory motion and rotatory motion.** *When every line in a body remains unchanged in direction as the body moves the motion is called pure translation.* The simplest example of pure translation is the motion of a car on a straight track. The most general case of pure translation is as follows: Grasp a stick at its middle and move the stick up and down and to and fro in any way whatever, but without turning the stick, as suggested by the curved dotted line in Fig. 58.

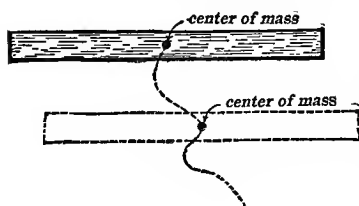


Fig. 58.

*When a certain line in a body remains stationary as the body moves, the motion is called pure rotation, and the stationary line is called the axis of rotation.* Rotation about a fixed axis is the simplest example of pure rotation. The most general case of

pure rotation is as follows: Consider a point in a body and imagine the body to be turned in any manner whatever but without moving the point.

**22. Center of mass of a body.**—Before proceeding to the discussion of the dynamics of translatory motion it is necessary to consider the question: When does a single unbalanced force\* produce pure translatory motion of the body on which it acts? and for the sake of simplicity let us consider this question from a purely experimental point of view.

A stick is held loosely between thumb and forefinger, struck with a hammer and allowed to fall. The force exerted on the stick by the hammer, while it lasts, is so large that the other

\* This single unbalanced force may, of course be the resultant of a set of forces.

forces (the force exerted on the stick by the supporting hand and the pull of gravity on the stick) are negligible in comparison; therefore the force exerted on the stick by the hammer may be thought of as the only force acting on the stick; this force may produce a combination of translatory and rotatory motion as indicated in Fig. 59, or it may produce pure translatory motion

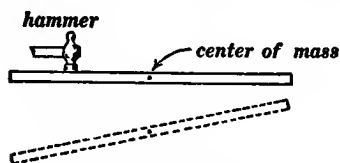


Fig. 59.

Hammer blow produces a combination of translation and rotation.

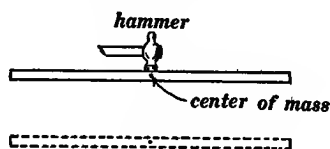


Fig. 60.

Hammer blow produces pure translation.

as indicated in Fig. 60.\* The center of mass of a body is the point at which a single force must be applied (through which the line of action of the single force must pass) to produce pure translation.

Take hold of a long slim stick at its middle (at its center of mass), using thumb and finger, and move the stick up and down and to and fro in any complicated manner, but without turning the stick. Closing one's eyes, it feels as if one had a very heavy piece of lead between thumb and finger, that is to say, the material of the stick seems to be concentrated between thumb and finger (at the center of mass of the stick); but if one twirls the stick even to a very slight extent it no longer feels like a piece of lead between thumb and finger! This experiment must be actually tried to be appreciated. The center of mass of a body is the point at which all of the material of the body may be thought of as concentrated in so far as the behavior of the body is concerned while it is performing purely translatory motion.

\* The difference between Fig. 59 and Fig. 60 may be made evident to an entire audience by calling attention to the noise produced when the stick strikes the floor. In Fig. 59 the stick strikes so as to produce a clattering noise, whereas in Fig. 60 the stick strikes the floor flat-wise and produces a sharp slam.

Throughout this chapter the single unbalanced force which acts on a body (or the *RESULTANT* of all the forces which act on the body) is understood to act at the center of mass of the body, thereby producing pure translatory motion.

**23. Velocity.** Let  $P$ , Fig. 61, be the position of a ball at a given instant, and let  $Q$  be the position of the ball  $t$  seconds later; then, however crooked the path over which the ball has actually traveled and however irregular the motion of the ball may have been, the quotient  $s/t$  is the *average velocity* of the ball during the  $t$  seconds,  $s$  being the length of straight line  $PQ$ . The highly important conception of *instantaneous velocity* is discussed in Art. 25.

**24. Rate of change of velocity; acceleration.**—Let the arrow  $v_1$  Fig. 62 represent the velocity of a ball at a given instant, and

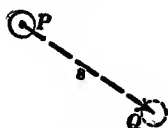


Fig. 61.

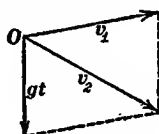


Fig. 62.

let the arrow  $v_2$  represent the velocity of the ball  $t$  seconds later. Then the arrow  $gt$  represents the velocity gained by the ball during the  $t$  seconds, and the quotient  $gt/t$  is the *average rate of gain of velocity* or *average acceleration* of the ball during the  $t$  seconds. The highly important conception of *instantaneous acceleration* is discussed in the next article.

**25. Two examples illustrating what is meant by instantaneous velocity and instantaneous acceleration.**—**Example 1.** A ball starts from the point  $O$  Fig. 63, and travels along the line  $OB$  so that the distance  $y$  of the ball from  $O$  is

$$y = bt^2 \quad (\text{i})$$

where  $b$  is a constant and  $t$  is the time in seconds that has elapsed since the ball started from  $O$ . What is the velocity of the

ball at the instant it reaches  $B$  and what is its acceleration at the same instant?

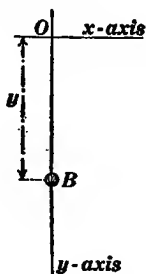


Fig. 63.

Showing position of moving ball  $t$  seconds after starting from  $O$ .

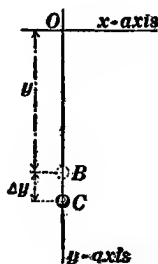


Fig. 64.

Showing position of moving ball  $t + \Delta t$  seconds after starting from  $O$ .

At instant  $t$  the ball is at  $B$  and  $\Delta t$  seconds later the ball is at  $C$ , and, inasmuch as equation (i) is assumed to give the distance traveled by the ball during any lapse of time whatever, it is evident that

$$y + \Delta y = b(t + \Delta t)^2 \quad (\text{ii})$$

Expanding  $(t + \Delta t)^2$ , equation (ii) becomes

$$y + \Delta y = bt^2 + 2bt.\Delta t + b(\Delta t)^2 \quad (\text{iii})$$

and, subtracting (i) from (iii) member by member, we get

$$\Delta y = 2bt.\Delta t + b(\Delta t)^2$$

whence

$$\frac{\Delta y}{\Delta t} = 2bt + b.\Delta t \quad (\text{iv})$$

Now, according to Art. 23,  $\Delta y/\Delta t$  is the average velocity of the ball during the time interval  $\Delta t$ , and the so-called *instantaneous velocity* of the ball at the given instant (the actual velocity of the ball when it is at  $B$ ) is its "average velocity" during an infinitely short interval of time which includes the given instant. Let us therefore consider how the average velocity  $\Delta y/\Delta t$  changes in value as the time interval  $\Delta t$  is chosen shorter and shorter.

It is evident from equation (iv) that  $\Delta y/\Delta t$  becomes more and more nearly equal to  $2bt$  as  $\Delta t$  approaches zero. In fact  $2bt$  is the limiting value of  $\Delta y/\Delta t$  as  $\Delta t$  approaches zero. This limiting value of  $\Delta y/\Delta t$  is usually represented by the complicated symbol  $\frac{dy}{dt}$ , and therefore we may write

$$\frac{dy}{dt} = 2bt, \quad (\text{v})$$

but the "average velocity" of the ball (the value of  $\Delta y/\Delta t$ ) during an infinitely short interval of time is by definition the actual or instantaneous velocity  $v$  of the ball at the given instant  $t$  so that we may write

$$v = \frac{dy}{dt} = 2bt \quad (\text{vi})$$

From this equation it is evident that  $v = 2bt$  is the velocity of the ball at the instant  $t$ , and  $\Delta t$  seconds later, when the velocity has increased to  $v + \Delta v$ , we have

$$v + \Delta v = 2b(t + \Delta t) \quad (\text{vii})$$

Therefore, subtracting equation (vi) from equation (vii) member by member, we get

$$\Delta v = 2b \cdot \Delta t$$

whence

$$\frac{\Delta v}{\Delta t} = 2b \quad (\text{viii})$$

Now, according to Art. 24,  $\Delta v/\Delta t$  is the *average acceleration* of the ball during the time  $\Delta t$ , and the *instantaneous acceleration* of the ball (actual acceleration) at the given instant  $t$  is the "average acceleration" during an infinitely short interval of time which includes the given instant. In the example under consideration, namely, when the distance  $y$  traveled by the ball is equal to  $bt^2$ , the average acceleration during the interval  $\Delta t$  is equal to  $2b$  however long or short the interval  $\Delta t$  may be [see equation (viii)]. That is to say, the acceleration in this

example is constant so that average acceleration and instantaneous acceleration are identical.\*

**Example 2.**—A ball travels steadily around a circle of radius  $r$ , making  $n$  revolutions per second; and it is required to find the velocity of the ball and its acceleration at the instant the ball passes a chosen point  $P$  (see Figs. 65 and 67.)

*Determination of velocity of ball.*—At the chosen instant the

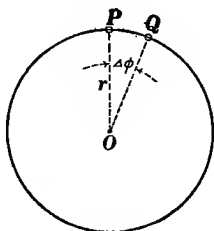


Fig. 65.

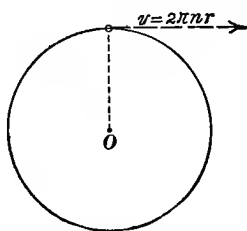


Fig. 66.

ball is at  $P$ , Fig. 65, and  $\Delta t$  seconds later the ball is at  $Q$ . The working out of this example is enormously simplified by thinking of the interval  $\Delta t$  as indefinitely short at the very beginning of the argument; therefore the infinitely short arc  $PQ$  may be thought of as a straight line, and since the angle  $\Delta\phi$  is indefinitely small the line  $PQ$  is at right angles to  $OP$ . The ball makes  $n$  revolutions per second or  $2\pi n$  radians per second (there being  $2\pi$  radians in a revolution). Therefore  $\Delta\phi = 2\pi n \cdot \Delta t$  radians; but  $\Delta\phi$  in radians is equal to  $PQ/r$ , so that  $PQ = 2\pi nr \cdot \Delta t$ ; and if we divide  $PQ$  by  $\Delta t$  we get the velocity  $v$  of the particle. Therefore  $v = 2\pi nr$ , and this velocity is in the direction shown in Fig. 66.

It may seem sufficient to say that the velocity of the ball is  $2\pi nr$  because  $2\pi nr$  is the distance traveled per second along the circumference of the circle; but the more lengthy argument is perhaps worth while.

\* Let the student work out the example in which the distance  $y$  traveled by the ball in  $t$  seconds is  $y = bt^3$  and show that instantaneous velocity  $v = \frac{dy}{dt} = 3bt^2$  and that instantaneous acceleration  $a = \frac{dv}{dt} = 6bt$ .

*Determination of acceleration.*—At the given instant the ball is at  $P$ , Fig. 67, and its velocity is  $v_1$ ; and  $\Delta t$  seconds later the

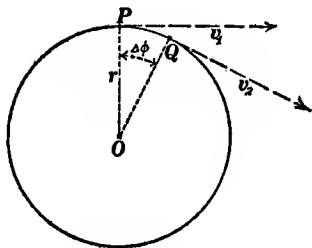


Fig. 67.

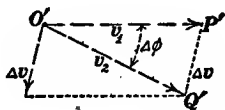


Fig. 68.

ball is at  $Q$  and its velocity is  $v_2$ . To find how much velocity has been gained by the ball during the interval  $\Delta t$  we must find how much velocity must be added to  $v_1$  to give  $v_2$ , remembering that velocities, like forces, are added by the "parallelogram law" as explained in Art. 2, Chapter I. Therefore let  $v_1$  and  $v_2$  be represented by lines drawn from the same point  $O'$  in Fig. 68. Then the arrow  $\Delta v$  represents the velocity which must be added to  $v_1$  to give  $v_2$ , that is, the arrow  $\Delta v$  represents the velocity gained by the ball during the interval  $\Delta t$ .

The time interval  $\Delta t$  being thought of as indefinitely small, the angle  $\Delta\phi$  is indefinitely small, the excessively short arc  $PQ$  may be thought of as a straight line, and the triangles  $OPQ$  and  $O'P'Q'$  are similar. Therefore, using  $v$  for the common numerical value of  $v_1$  and  $v_2$ , we have

$$\frac{PQ}{OP} = \frac{P'Q'}{O'P'} \quad \text{or} \quad \frac{v \cdot \Delta t}{r} = \frac{\Delta v}{v}$$

whence

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

Therefore, writing  $a$  for the acceleration  $\Delta v/\Delta t$ , we have

$$a = \frac{v^2}{r} \quad \text{or} \quad a = 4\pi^2 n^2 r$$

and the gained velocity  $\Delta v$  in Fig. 68 is parallel to  $PO$  in Fig. 67 (angle  $\Delta\phi$  being indefinitely small) so that *the ball is at each instant gaining velocity towards the center of its circular orbit at a rate equal to  $v^2/r$  or to  $4\pi^2n^2r$ .*

**26. Differentiation.**—Having given  $y = bt^2$ , where  $b$  is a constant, the finding of an expression for  $\frac{dy}{dt}$ , as exemplified in Art. 25, is called *differentiation*, and  $\frac{dy}{dt}$ , which is equal to  $2bt$  when  $y = bt^2$ , is called the *rate of change of  $y$*  or the *derivative of  $y$  with respect to  $t$* . Likewise, having given  $T = kx^3$ , where  $k$  is a constant, the finding of an expression for  $\frac{dT}{dx}$  is called *differentiation* and  $\frac{dT}{dx}$  is called the *derivative of  $T$  with respect to  $x$* . Example 1 of Art. 25 is called *algebraic differentiation*, and example 2 of Art. 25 is called *geometric differentiation*.

**Meaning of a derivative.**—Let  $T = kx^3$ , then it is easy to show that  $\frac{dT}{dx} = 3kx^2$  which means that if  $x$  grows,  $T$  must always grow  $3kx^2$  times as fast. The equation  $\frac{dT}{dx} = 3kx^2$  expresses the law of growth of  $T$ .

**The setting up of a derivative.**—The area  $A$  in Fig. 69 evidently depends on  $x$ ; if  $x$  grows larger the area  $A$  also grows larger, and even though we may not know the algebraic expression for  $A$  in terms of  $x$ , it is easy to establish an expression for the derivative of  $A$  with respect to  $x$ , or, in other words, *to express the law of growth of  $A$  when  $x$  grows steadily.*

Let the curve  $cc$  in Figs. 69 and 70 be a parabola, a curve whose equation is:

$$y = bx^2 \tag{i}$$

where  $b$  is a constant. The narrow strip  $\Delta A$  in Fig. 70 is the increase of the shaded area  $A$  when  $x$  increases by the amount  $\Delta x$  and the limiting value of the quotient  $\Delta A/\Delta x$  as  $\Delta x$  ap-

proaches zero is called the derivative of  $A$  with respect to  $x$  and it is represented by the complicated symbol  $\frac{dA}{dx}$ . If we think of  $\Delta x$  as being indefinitely small in Fig. 70, the area  $\Delta A$  may be thought of as an indefinitely narrow *rectangle* whose

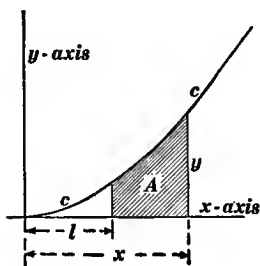


Fig. 69.

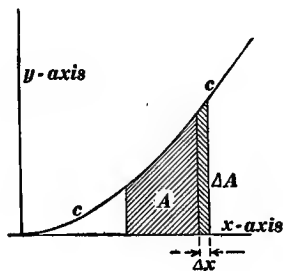


Fig. 70.

altitude is  $y$  and whose width is  $\Delta x$ , and therefore when  $\Delta x$  is indefinitely small  $\Delta A = y \Delta x$ , or  $\frac{\Delta A}{\Delta x} = \frac{dA}{dx} = y$ . Consequently, using  $bx^2$  for  $y$ , according to equation (i), we have

$$\frac{dA}{dx} = bx^2 \quad (\text{ii})$$

This is an example of what is called *setting up a derivative*, and this sort of thing is extremely important in the application of calculus to physics. The setting up of derivatives is further exemplified in Arts. 36, 53 and 63.

**27. Integration.**—The law of growth of the area  $A$  in Fig. 69 is expressed by equation (ii) which means that  $A$  always grows  $bx^2$  times as fast as  $x$ . How are we to determine the total growth of  $A$  while  $x$  is increasing from any given value  $x = l$  to the value  $x$  which is shown in Fig. 69? The argument which leads to the desired growth of  $A$  is called *integration*, and the most important part of this argument is the following proposition:

John saves money always at the same rate as Henry, or, symbolically expressed,  $\frac{dJ}{dt} = \frac{dH}{dt}$ , where  $J$  is the amount of John's money and  $H$  is the amount of Henry's money. Everyone understands that if John and Henry always save at the same rate *neither can ever gain on the other*, that is we must have  $J - H = \text{a constant}$ , or  $J = H + \text{a constant}$ .

This proposition, which is so easily understood as referring to John and Henry, may be stated in general by saying **any two quantities  $J$  and  $H$  whose derivatives are equal have a constant difference, or  $J = H + C$ , where  $C$  is a constant.**

(a) The first step in "integrating" the equation

$$\frac{dA}{dx} = bx^2 \quad (\text{i})$$

is to find an expression whose derivative is the same as  $\frac{dA}{dx}$  (an expression whose derivative is equal to  $bx^2$ ). In some cases this first step is difficult because it is essentially the making of a discovery. Consider the expression  $Z = cx^3$ . Proceeding as in Example 1 of Art. 25 we find that  $\frac{dZ}{dx} = 3cx^2$ . Therefore if we make  $3c = b$  or  $c = b/3$  we get  $Z = \frac{1}{3}bx^3$  and  $\frac{dZ}{dx} = bx^2$ . That is to say, we have an expression  $Z = \frac{1}{3}bx^3$  whose derivative is the same as  $\frac{dA}{dx}$ .

(b) The second step is to make use of the above proposition. Knowing that  $Z = \frac{1}{3}bx^3$  has the same derivative as  $A$ , we know that  $A$  must be equal to  $Z$  plus a constant; that is, we have

$$A = \frac{1}{3}bx^3 + C \quad (\text{ii})$$

where  $C$  is an undetermined constant which is called the *constant of integration*.

(c) The third step is to determine the value of  $C$  from a

knowledge as to the value of  $A$  for some particular value of  $x$ . From Fig. 69 it is evident that  $A = 0$  when  $x = l$ . Therefore putting  $A = 0$  and  $x = l$  in equation (ii) we get  $0 = \frac{1}{3}bl^3 + C$  or  $C = -\frac{1}{3}bl^3$ ; and, using this value for  $C$ , equation (ii) becomes

$$A = \frac{1}{3}bx^3 - \frac{1}{3}bl^3 \quad (\text{iii})$$

#### PROBLEMS.

**56.** The distance  $s$  in feet traveled by a body in  $t$  seconds is  $s = at^3$ . Find an expression for the velocity of the body and for the acceleration of the body at the instant  $t$ .

**57a.** Let the curve  $cc$  in Fig. 69 be a straight line  $y = bx$ . Find by integration an expression for the area  $A$ .

**57b.** Let the equation of the curve  $cc$  in Fig. 69 be  $y = kx^3$ . Find an expression for the area  $A$ .

**57c.** A man saves money continuously at a rate  $\frac{dM}{dt} = kt^2$ , which evidently means that his rate of saving is zero at the beginning when  $t = 0$ . After 1000 days his rate of saving is, let us say, five dollars per day. What is the value of  $k$ ? How much money is saved during the period of 1000 days?



**28. How does a body behave when acted upon by an unbalanced force?**—It gains velocity in the direction of the force, and the amount of velocity gained during an interval of time  $\Delta t$  is proportional to  $\Delta t$ , it is also proportional to the force, and it is inversely proportional to the mass of the body; or, in other words, the *acceleration*  $a$  produced by an unbalanced force  $F$  is in the direction of the force, it is proportional to the force, and it is inversely proportional to the mass  $m$  of the body.\*

That is to say,  $a$  is proportional to  $F/m$  or  $a = k \left( \frac{F}{m} \right)$ , where

\* This fact was discovered by Sir Isaac Newton, and it is usually referred to as Newton's second law of motion.

$k$  is a constant, or

$$F = \frac{1}{k} ma \quad (i)$$

Now, if we adopt as our unit of force that force which will produce unit acceleration when it acts (as an unbalanced force) on a body of unit mass, then  $F = 1$  when  $m = 1$  and  $a = 1$ , so that the factor  $(1/k)$  is equal to unity, and equation (i) becomes

$$F = ma \quad (I)$$

**29. The c g.s. unit of force** is called the *dyne* and it is the force which will accelerate a one-gram body at the rate of one centimeter per second per second. The pull of the earth on a one gram body at Washington, D. C., produces an acceleration of 980.100 centimeters per second per second. Therefore the pull of the earth on a one gram body at Washington is equal to 980.100 dynes. The pull of the earth on a one-gram body at Hammerfest (70° 40' north latitude) it is 982.580 dynes.

**30. The slug as a unit of mass.**—The pull of the earth on a one pound body at London produces an acceleration of 32.174 feet per second per second. Therefore this same force (a one-“pound” pull) would produce an acceleration of one foot per second per second if it were to act as an unbalanced force on a 32.174-pound body. This amount of material is called a *slug*. Mass in pounds must be divided by 32.174 (a pure number) to get mass in slugs.

**31. C.g.s. units and f.s.s. units.**—Two systems of units are used in this text, namely, the **c.g.s. system** which is based on the *centimeter* as the unit of length, the *gram* as the unit of mass and the *second* as the unit of time, and the **f.s.s. system** which is based on the *foot* as the unit of length, the *slug*\* as the unit of

\* As a matter of fact the “pound” of force is one of the fundamental units in this system and the slug is a derived unit. It is best, however, to speak of this system as the foot-slug-second system because the word pound has two meanings, as when we speak of 10 pounds of sugar or 10 “pounds” of pull.

mass, and the *second* as the unit of time. **Units of either system may be used in any formula in mechanics (including hydrostatics and hydraulics) when the formula is in its simplest form; and all formulas are given in their simplest form in this text; BUT A MIXTURE OF UNITS LEADS TO HOPELESS CONFUSION.**

TABLE.

Quantity.	C.G.S. unit.	F.S.S. unit.
length	centimeter	foot
mass	gram	slug
time	second	second
area	square centimeter	square foot
volume	cubic centimeter	cubic foot
density	gram per cubic centimeter	slug per cubic foot
velocity	centimeter per second	foot per second
acceleration	cm. per second per second	foot per second per second
spin-velocity	radian per second	randian per second
spin-acceleration	radian per second per second	radian per second per second
force	dyne	"pound"
work or energy	cm.-dyne or erg	foot-"pound"
power	erg per second	foot-"pound" per second
torque*	dyne-centimeter	"pound"-foot
moment of inertia	gram-(centimeter) <sup>2</sup>	slug-(foot) <sup>2</sup>
hydrostatic pressure	dyne per square centimeter	"pound" per square foot

The English commercial unit of mass is the *pound*; mass in pounds must be divided by the pure number 32.174 to give mass in slugs.

**32. Discussion of mass and weight.**—The result obtained when a body is "weighed" on a balance scale is called the **MASS** of the body, and it is properly expressed in grams or pounds or slugs. The word pound as here used is the *sugar-pound*. What is technically called the mass of a body is called the "weight" of the body in commerce. *Technically the WEIGHT of a body means the force with which the earth pulls on the body*, and it is properly expressed in dynes or "pounds." The word "pound" as here used is the *pull-pound*.

\* The unit of torque is of course a very different thing physically from the unit of work or energy, although torque is *force times distances* and work also is *force times distance*. It is well to form the habit, therefore, of speaking of foot-"pounds" of work or energy and of "pound"-feet of torque.

Let  $w$  be the weight of a body (gravity pull on the body),  $m$  the mass of the body and  $g$  the acceleration produced by the force  $w$  (the local acceleration of gravity). Then

$$w = mg \quad (2)$$

$w$  in dynes is equal to  $m$  in grams multiplied by the local acceleration of gravity in centimeters per second per second.

$w$  in "pounds" is equal to  $m$  in slugs multiplied by the local acceleration of gravity in feet per second per second.

The most important thing to remember concerning equation (2) is that  $w$  is *not* what your coal-man calls the "weight" of the body and  $w$  is **NOT** determined by the balance scale.\*

**33. Example of accelerated motion.**—(1) A 2,000,000-pound train is started and brought up to a velocity of 60 feet per second in 240 seconds on a straight, level track. The average acceleration of the train during the 240 seconds is 0.25 foot per second per second, and the average value in "pounds" of the unbalanced force which accelerates the train is found by multiplying the average acceleration in feet per second per second by the mass of the train in slugs which gives 15,500 "pounds." The backward drag (friction drag) on a 2,000,000-pound train is about 10,000 "pounds," so that the draw-bar pull of the locomotive must be about 25,500 "pounds."

(2) A 200,000-pound locomotive exerts a draw-bar pull of

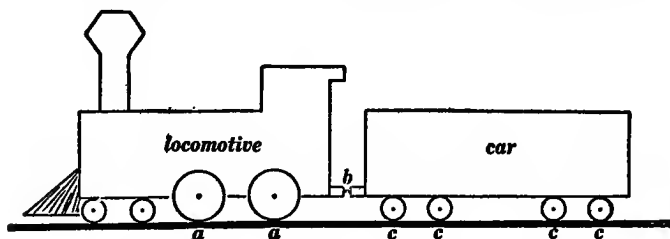


FIG. 71.

\* Of course the *weight* of a body in "pounds" is very nearly equal *numerically* to the *mass* of the body in pounds; so is 1000 dollars very nearly equal *numerically* to 1001 feet! Numerical precision has very little to do with precision of thinking.

25,500 "pounds" while it accelerates a train (and itself) at the rate of 0.25 foot per second per second. With how much force must the locomotive drive wheels push backwards on the rails at the points *aa* in Fig. 71? The unbalanced forward push on the locomotive must be 1,550 "pounds" according to equation (1), which means that the forward push of the rails at *aa* must be 1,550 "pounds" greater than the backward pull of the train at *b* (25,500 "pounds"). Therefore the forward push on rims of drive wheels at *aa* is 27,050 "pounds."

(3) An elevator cage starts and gains full speed of 8 feet per second upwards in 1.5 seconds. With what average force must the cage platform push upwards on the feet of a 160-pound man while the car is starting upwards? The average upward acceleration of the cage while starting is  $5\frac{1}{3}$  feet per second per second so that by using equation (1) we find that the average unbalanced upward force acting on the man's body must be 26.5 "pounds." But the downward pull of gravity on the man is very nearly 160 "pounds"; therefore the upward push of the platform must be 186.5 "pounds."

(4) When a heavy ball is thrown through the air at low velocity, the friction of the air is negligible; the only appreciable force acting on the ball is the constant downward pull of gravity on the ball, and this constant downward pull causes the ball to

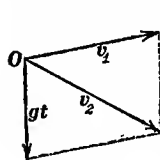


FIG. 72.

gain downward velocity at a constant rate  $g$  (about 32 feet per second per second). At a given instant the velocity of the ball is as represented by the arrow  $v_1$  in Fig. 72;  $t$  seconds later the ball has gained downward velocity amounting to  $gt$ , and its velocity is then the

vector sum of  $v_1$  and  $gt$  as represented by the arrow  $v_2$  in Fig. 72.

It is highly instructive to consider this example from a slightly different point of view, as follows: Suppose the initial velocity  $v_1$  to be equivalent to an upward velocity of 60 feet per second and a horizontal velocity of 100 feet per second. After 3 seconds

the ball will have gained 96 feet per second of downward velocity ( $gt$ ), and its upward velocity will then be *minus* 36 feet per second (which is, of course, a downward velocity); but the horizontal velocity of the ball is not altered by the vertical pull of gravity.

## PROBLEMS.

58. A 700,000-pound train is started and brought up to a velocity of 50 miles per hour in 110 seconds on a straight level track. The backward drag on the train due to friction is 4000 "pounds." Find the average acceleration of the train and find the average draw-bar pull of the locomotive.

59. A 150,000-pound locomotive exerts a draw-bar pull of 12,000 "pounds" while accelerating a train at the rate of 0.6 foot per second per second on a straight level track. Find forward push of rails on rims of locomotive drive wheels.

60. The maximum tractive force (backward push of rims of drive wheels on rails) that can be exerted by a locomotive is, let us say, one-tenth of the *weight* of the locomotive. The mass of the locomotive is 300,000 pounds. Find the maximum draw-bar pull that can be exerted by the locomotive while it is accelerating a train at the rate of 0.4 foot per second per second on a straight level track.

*Note.*—What is the *weight* of a 300,000-pound locomotive? If it were in London its *weight* would be exactly 300,000 "pounds," and since the weight of a body varies only a fraction of one per cent at different places on the earth it is sufficient for most practical purposes to ignore the variation altogether. In this and the following problems, therefore, take weight in "pounds" as equal to mass in pounds. The exact weight of a body is considered in problem 64.

61. The maximum tractive effort exerted by a 300,000-pound locomotive is, let us say, 37,500 "pounds." Find maximum acceleration which the locomotive can produce up a one per cent. grade when hitched to a 1,500,000-pound train, assuming total friction drag on locomotive and train to be 7,500 "pounds." A one per cent. grade rises one foot in a horizontal distance of 100 feet.

62. An elevator cage has a mass of 2500 pounds and it gains

full speed of 10 feet per second upwards or downwards in 1.25 seconds after starting. Assuming that acceleration is constant while it lasts, and assuming that cage is subject to no air friction or guide friction, find (a) Tension of cable while car is starting downwards, and (b) Tension of cable while cage is moving upwards or downwards at full speed.

63. A spring scale registers 10 "pounds" when a 10-pound body is hung upon it. What would the scale register in an elevator cage which has an upward acceleration of 8 feet per second per second? In an elevator cage which has a downward acceleration of 8 feet per second per second?

64. The acceleration of gravity at London is 32.174 feet per second per second. Find the exact *weight* in "pounds" of a 1000-pound body at a place where the acceleration of gravity is 31.823 feet per second per second.

*Note.*—Reduce the thousand pounds to slugs by dividing by 32.174, and multiply mass in slugs by 31.823 feet per second per second to get weight in "pounds."

65. A cord is strung over a pulley. On one end of the cord a 10-pound body is hung, and on the other end of the cord a 12-pound body is hung. Neglecting mass and weight of cord, neglecting mass of pulley, and neglecting friction, find acceleration of each body and tension of cord.

*Note.*—See note to problem 60. In this problem *two bodies* are involved. Apply the equation  $F = ma$  to each body separately. How do you know that both bodies have the same amount of acceleration? One is of course accelerated downwards and the other is accelerated upwards.

66. A cord is tied to a 25-pound block on a level table, and a 15-pound body is hung on the cord after it passes over a pulley at the edge of the table. Neglecting mass and weight of cord, and neglecting mass and friction of pulley, but taking coefficient of friction between block and table as 0.12, find acceleration of block and tension of cord.

*Note.*—Two bodies are involved in this problem. Apply the equation  $F = ma$  to each body separately.

**34. Uniformly accelerated motion.**—When a constant\* unbalanced force acts on a body the body gains velocity at a constant\* rate. Such a body is said to perform *uniformly accelerated motion*. A body which is falling freely under the action of the constant pull of gravity is, in so far as the friction of the air is negligible, an example of uniformly accelerated motion. If the body is moving vertically we have the *simple falling body*, if the body is thrown like a ball we have what is called a *projectile*.

Let  $g$  be the constant rate of gain of velocity by a falling body, then  $gt$  is the amount of downward velocity gained in  $t$  seconds. That is

$$\text{gain of velocity} = gt \quad (\text{i})$$

Let  $v_1$  be the velocity of the falling body at the beginning of the  $t$  seconds, then  $v_1 + gt$  is the velocity at the end of the  $t$  seconds.

Now the average value during any given time of any quantity which changes at a constant rate is equal to half the sum of the values at the beginning and end of the time,† so that the average velocity of a falling body during  $t$  seconds is half the sum of  $v_1$  and  $v_1 + gt$ . That is, the average velocity is  $v_1 + \frac{1}{2}gt$ , and the distance  $s$  traveled by the falling body is equal to the average velocity multiplied by the time. Therefore:

$$s = v_1 t + \frac{1}{2}gt^2 \quad (\text{ii})$$

If the velocity  $v_1$  at the beginning is upward it is to be con-

\* Constant in value and unchanging in direction.

† This proposition may be understood as follows: Let the constantly increasing velocity of a falling body be represented by the ordinates of a curve as shown in Fig. 73. This "curve" is a straight line, and the average ordinate of any portion  $AB$  of this straight line is equal to  $\frac{1}{2}(v_1 + v_2)$ .

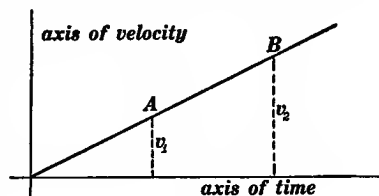


Fig. 73.

sidered as negative, and  $s$  is positive or negative according as the body is *below* or *above* its starting point after  $t$  seconds.

**35. Examples of uniformly accelerated motion.**—(1). A 2,200,000-pound train (including locomotive) moving at a velocity of 80 feet per second travels 2400 feet in being brought to rest by the brakes on a straight level track. *Assuming that the deceleration (negative acceleration) of the train is constant*, it is required to find the backward force acting on the train. If the deceleration is constant the average velocity of the train while stopping is half the sum of its initial and final velocities, or  $\frac{1}{2}$  (80 feet per second + 0), which is 40 feet per second; and since the average velocity while stopping is 40 feet per second it will take 60 seconds for the train to travel over the stretch of 2400 feet. Therefore the train loses its entire velocity of 80 feet per second in 60 seconds, its deceleration is  $1\frac{1}{3}$  feet per second per second, and the backward force required to produce this deceleration, as found by using equation (1), is 91,200 "pounds."

**Remark.**—Almost any problem in uniformly accelerated motion or any problem in falling bodies can be solved by an argument like the above; the beginner should never use formulas for such problems.

(2) A projectile leaves a gun at a velocity of 2500 feet per second in a direction  $30^\circ$  above the horizontal. Where is the projectile after 10 seconds, neglecting\* the friction of the air? The initial velocity of the projectile is equivalent to a horizontal velocity of 2165 feet per second and an upward velocity of 1250 feet per second. Therefore after 10 seconds the horizontal distance from the gun to the projectile is 21,650 feet. During 10 seconds the projectile gains 322 feet per second of downward velocity, its upward velocity at the end of the 10 seconds is 928 feet per second, its average upward velocity during the 10 seconds is  $\frac{1}{2}$  (1250 + 928) or 1089 feet per second, and its vertical distance from the gun at the end of 10 seconds is 10,890 feet.

(3) What is the *range* of the projectile in the previous example,

\* The friction of the air is *not* negligible by any means.

that is to say, how far is the projectile from the gun when it reaches the level of the gun on its downward flight, resistance of air being assumed to be zero? During the whole time of flight the average upward velocity is zero because the total vertical travel amounts to zero. Therefore the final vertical velocity  $v_2$  must be such that  $\frac{1}{2} (1250 - v_2)$  equals zero. Consequently  $v_2 = -1250$  feet per second so that the total gain of downward velocity is 2500 feet per second. At the rate of 32.2 feet per second per second (rate of gain) it takes 77.6 seconds to gain 2500 feet per second. Therefore the projectile is in the air for 77.6 seconds and traveling at a constant horizontal velocity of 2165 feet per second so that the total horizontal travel is 168,000 feet.

### 36. Calculus discussion of uniformly accelerated motion.—

Let  $x$  and  $y$  be abscissa and ordinate, respectively, of a projectile at an instant  $t$  seconds after it leaves the muzzle of the gun as indicated in Fig. 74. Let  $h$  be the horizontal velocity

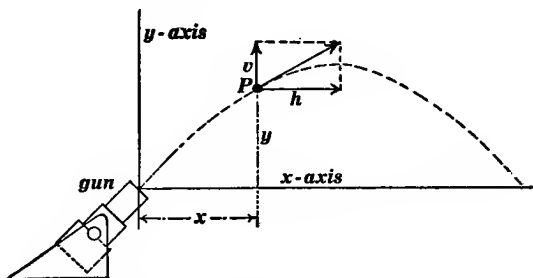


Fig. 74.

$P$  is the position of the projectile  $t$  seconds after leaving the muzzle of the gun;  $h$  is the horizontal velocity of the projectile and  $v$  is its vertical velocity at the instant  $t$ .

of the projectile as it leaves the muzzle of the gun. Air resistance, only, changes the value of  $h$ , and if we assume air resistance to be zero\* the value of  $h$  will be constant, and therefore:

$$x = ht \quad (i)$$

\* Air resistance is in fact very large for high-velocity projectiles, and by no means really negligible. See problem 73.

The calculus method for finding an expression for  $y$  is very meager in its purely formal aspects and for this rather absurd reason it is very difficult for the beginner to understand. Therefore the following discussion is purposely made somewhat elaborate.

(a) *Setting up an expression for the derivative  $\frac{dy}{dt}$ .*—Air resistance being neglected the projectile gains downward velocity at a constant rate  $g$ , and therefore the amount of downward velocity gained in  $t$  seconds is  $gt$ . Let  $v_0$  be the initial upward velocity of the projectile, then the upward velocity  $v$  after  $t$  seconds is

$$v = v_0 - gt$$

But  $v$  is the limiting value of  $\Delta y/\Delta t$  at any instant, and this limiting value of  $\Delta y/\Delta t$  is the derivative  $\frac{dy}{dt}$ . Therefore we have

$$\frac{dy}{dt} = v_0 - gt$$

where  $v_0$  and  $g$  are constants.

(b) *Discovery of a known expression or function which has the same derivative as  $y$ .*—Consider the function

$$z = at + bt^2$$

where  $a$  and  $b$  are constants. Using the method which is exemplified in Art. 25, we get

$$\frac{dz}{dt} = a + 2bt$$

Therefore, if we choose  $a = v_0$  and  $b = -\frac{1}{2}g$ , we have

$$z = v_0 t - \frac{1}{2}gt^2 \quad (\text{ii})$$

and

$$\frac{dz}{dt} = v_0 - gt \quad (\text{iii})$$

That is to say, we have the known function  $z = v_0 t - \frac{1}{2}gt^2$

whose derivative is the same as the known derivative of  $y$ . Therefore:

(c) *Using the proposition which is given in Art. 27*, we know that  $y$  must be equal to  $z$  plus a constant; that is

$$y = v_0 t - \frac{1}{2} g t^2 + C \quad (\text{iv})$$

(d) *Determination of C*.—The value of  $y$  is known to be zero when  $t = 0$ . Therefore, substituting  $y = 0$  and  $t = 0$  in (iv) we get  $C = 0$ , and consequently the desired expression for  $y$  is

$$y = v_0 t - \frac{1}{2} g t^2 \quad (\text{v})$$

**Remark.**—The above discussion is an example of a very important type of calculus problem. To determine by integration a desired expression for some quantity  $y$  four steps are necessary, namely, (a) To set up an expression for the derivative  $\frac{dy}{dt}$  (or for the derivative  $\frac{dy}{dx}$ ), (b) To discover a function  $z$  which has the same derivative, (c) To write down the most general possible expression for  $y$ , namely,  $y = z + C$  where  $C$  is an undetermined constant of integration, so-called, and (d) To determine the value of  $C$  from a known value of  $y$  for a particular value of  $t$  (or  $x$ ).

#### PROBLEMS.

67. A falling ball passes a given point at a velocity of 12 feet per second. How far below the point is the ball after 5 seconds? How far does the ball fall during the fifth second after passing the given point? Air friction neglected.

68. Show that the velocity gained by a body is  $\sqrt{2gs}$  where  $s$  is the distance fallen, and when the body starts to fall with zero velocity.

69. A heavy iron ball is tossed at a velocity of 20 feet per second in a direction  $30^\circ$  above the horizontal. What are its horizontal and vertical distances from the starting point after 0.75 second? Air friction neglected.

**70.** A 10-pound hammer, moving at a velocity of 16 feet per second, strikes a spike, and the spike is pushed one-half an inch into the beam into which it is being driven. Assuming hammer to lose velocity at a constant rate while stopping, find (a) Time required to stop, (b) Average deceleration while stopping, and (c) Average force exerted on the spike by stoppage of hammer.

**71.** A 200-pound cannon ball having a velocity of 2,400 feet per second penetrates a clay bank to a depth of 12 feet in being stopped. Assuming cannon ball to lose velocity at a constant rate, find (a) Time required to stop, (b) Average deceleration while stopping, and (c) Average force with which the clay bank retards the ball.

**72.** A train starts from station *A* and is accelerated at the rate of  $\frac{3}{4}$  of a mile per hour per second for 40 seconds. The train then runs at full speed of 30 miles per hour, and then it is decelerated at the rate of  $1\frac{1}{2}$  miles per hour per second until it stops at station *B*. The distance from station *A* to station *B* is 2 miles. Find (a) distance traveled while accelerating, (b) distance traveled while decelerating, (c) distance traveled at full speed, and (d) average velocity of whole run. Ans. (a)  $\frac{1}{6}$  mile; (b)  $\frac{1}{12}$  mile; (c)  $1\frac{3}{4}$  miles; (d) 26.67 miles per hour.

**73.** A 10-inch projectile ranges 17,000 yards when fired at  $16^{\circ} 10'$  elevation with a muzzle velocity of 2250 feet per second. A 16-inch projectile exactly similar in form and fired at the same elevation with same muzzle velocity ranges 20,068 yards. (a) What per cent. of the range in vacuum is lost in each case? (b) Why is the loss greater in the one case than in the other case? Explain.

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**37.** The kinetic energy of a body of mass *m* moving at velocity *v* is

$$W = \frac{1}{2}mv^2 \quad (3)$$

The kinetic energy *W* is expressed in ergs when *m* is in grams and *v* in centimeters per second.

The kinetic energy  $W$  is expressed in foot-"pounds" when  $m$  is in slugs and  $v$  is in feet per second.

To establish equation (3) it is sufficient to show that an amount of work equal to  $\frac{1}{2}mv^2$  must be done on a body of mass  $m$  to get it moving at velocity  $v$ , friction being zero. Imagine a constant force  $F$  (unbalanced) to act on the body. Then, according to Art. 28, we have

$$F = ma \quad (\text{i})$$

The acceleration  $a$  will be constant and the amount of velocity gained in  $t$  seconds will be

$$v = at \quad (\text{ii})$$

The distance  $s$  traveled by the body (which starts with zero velocity) will be

$$s = \frac{1}{2}at^2 \quad (\text{iii})$$

and the amount of work done,  $W = Fs$ , is

$$W = \frac{1}{2}ma^2t^2 \quad (\text{iv})$$

but  $at = v$  so that  $a^2t^2 = v^2$  and therefore

$$W = \frac{1}{2}mv^2$$

**38. Momentum.**—The product  $mv$  is called the momentum of a body,  $m$  being the mass of the body and  $v$  being its velocity. The idea of momentum is useful in the discussion of the following problem: A bullet of mass  $m$  moving at velocity  $v$  strikes a suspended body of mass  $M$ . Find the velocity  $V$  of the suspended body after the impact of the bullet. See Fig. 75.

Consider any infinitely short interval of time  $\Delta t$  while the bullet is burying itself in the block  $B$ . During this time the

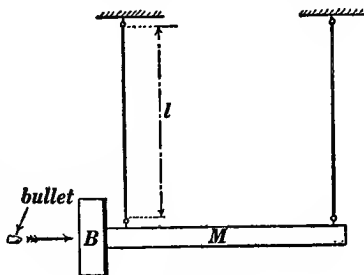


Fig. 75.

bullet exerts on the suspended body a definite force  $F$  which accelerates the suspended body, and the suspended body exerts an equal and opposite force on the bullet which decelerates the bullet, and therefore, according to Art. 28, we must have:

$$F = MA \text{ and } F = ma$$

or

$$MA = ma \quad (i)$$

where  $A$  is the acceleration of the suspended body (rate at which it is gaining velocity) and  $a$  is the deceleration of the bullet (rate at which it is losing velocity). Multiply both members of equation (i) by the short interval of time  $\Delta t$  and we get:

$$MA.\Delta t = ma.\Delta t$$

or

$$M.\Delta V = m.\Delta v \quad (ii)$$

where  $\Delta V$  is the velocity gained by  $M$  and  $\Delta v$  is the velocity lost by  $m$  during the time  $\Delta t$ , or  $M.\Delta V$  is the momentum gained by  $M$  and  $m.\Delta v$  is the momentum lost by  $m$ . Therefore equation (ii) being interpreted means that every bit of momentum lost by the bullet is gained by the suspended body so that *the total momentum of bullet and suspended body after impact of bullet is the same as at was before*. But the bullet, only, has momentum before the impact and its momentum is  $mv$ , whereas bullet and body (combined mass  $m + M$ ) move along as one body at velocity  $V$  after impact so that the total momentum after impact is  $(m + M)V$ . Therefore we have

$$mv = (m + M)V$$

or

$$V = \frac{m}{m + M} \cdot v \quad (iii)$$

**The principle of the conservation of momentum.**—The above example of bullet and suspended body (Fig. 75) is an illustration of a very important general principle, namely, *the mutual force action of any two bodies never alters their combined momentum, or the mutual force action of any number of bodies never alters*

*their combined momentum*, or the combined momentum of any system of bodies is constant if none of the bodies is acted on by forces originating outside of the system.

**39. The problem of the accelerated elevator cage.**—An elevator cage has a mass of 3000 pounds, its center of mass is at the point  $C$ , Fig. 76, the dimensions of the cage are as indicated in the figure, and the coefficient of friction  $a$  between cage and guides is 0.1. Find tension of cable  $P$  and find horizontal forces  $R$  and  $L$  exerted on the cage by the guides when the cage is moving upwards and being accelerated upwards at the rate of 8 feet per second per second.

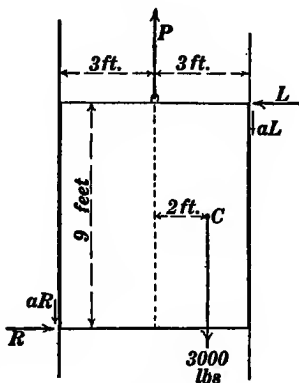


Fig. 76.

Pure translatory motion, only, is being produced and therefore all the forces which act on the cage must be together equivalent to a single upward force acting at  $C$  which means that the combined torque actions of all the forces about  $C$  must be zero. This condition gives one equation which is easily formulated.

The car has no horizontal acceleration and therefore the force  $R$  must be equal to the force  $L$ . This gives another equation.

The downward forces acting on the car are  $aL$ ,  $aR$  and the downward pull of gravity which we will take as 3000 "pounds"; and the upward force acting on the car is  $P$ . Therefore  $P - 3000$  "pounds"  $- aL - aR$  is the unbalanced upward force acting on the cage, and it must be equal to  $ma$  where  $m$  is the mass of the cage in slugs and  $a$  is its upward acceleration.

#### PROBLEMS.

**74.** A steamship has a mass of 25,000 tons. What is the kinetic energy of the ship at a speed of 18 miles per hour? Express the result in foot-"pounds" and in horse-power-hours.

*Note.*—Take one ton equal to 2,000 pounds.

75. The rim of the flywheel of a metal punch press has a mass of 560 pounds. What must be the initial velocity of the rim in feet per second in order that the press may exert an average force of 72,000 "pounds" while moving a distance of one inch, and reduce the velocity of the rim to 70 per cent. of its initial value? Assume the entire rim to have the same velocity and ignore the kinetic energy of moving spokes and hub.

76. A 16-inch seacoast gun has a mass of 130 long tons and its projectile has a mass of 2400 pounds. The muzzle velocity of the projectile is 2250 feet per second, and let us assume that the mean velocity of the powder gases as they leave the gun is 6000 feet per second, the total mass of the powder gases being 500 pounds. Furthermore let us imagine the gun to be entirely free to move backwards when it is fired. What would the recoil velocity of the gun be? What would the kinetic energy of the gun be? Kinetic energy of projectile? Kinetic energy of powder gases?

77. A 22-caliber bullet having a mass of 1.8 grams is shot into the block  $B$  in Fig. 75, mass of suspended body being 2000 grams. The length  $l$  of the suspending threads is 250 centimeters, and the block  $B$  pushes a light slider 16.5 centimeters along a horizontal meter stick after the bullet strikes  $B$ . Calculate velocity of suspended body produced by impact of bullet (take acceleration of gravity as 980 centimeters per second per second) and calculate velocity of bullet.

78. Find tension of cable in Fig. 76, when cage is moving downwards and has an upward acceleration of 8 feet per second per second, coefficient of friction  $\mu$  being 0.1.

*Note.* Consider all of the forces which act on the elevator cage in Fig. 76. Why is their combined torque action about an axis passing through  $C$  equal to zero? Because the forces are all together equivalent to a single force acting at  $C$  as explained in Art. 22. Is the combined torque action of all the forces in Fig. 76 about any axis whatever equal to zero? Certainly not.

79. Find tension of cable in Fig. 76, when cage is moving downwards and has a downward acceleration of 8 feet per second per second, coefficient of friction  $\mu$  being 0.1.

80. The distance between front and rear axles of a 1600-pound automobile is 7 feet and the center of mass of the automobile is 2 feet above the ground and 3 feet back of the front axle. Find the upward push of the ground on the front wheels and the upward push of the ground on the rear wheels:—(a) When the automobile is starting on a level road with an acceleration of 2 feet per second per second, and (b) When the automobile is stopping on a level road with a deceleration of 8 feet per second per second.

40. **Translatory motion in a circle.**—Imagine the stick which is shown in Fig. 58 to be moved so that the center of mass of the stick describes a circle of radius  $r$ , but without turning the stick in any way, as suggested by Fig. 77. The stick is then said to

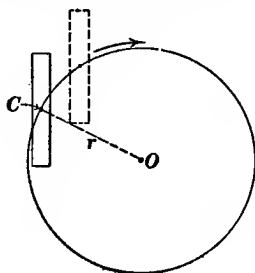


Fig. 77

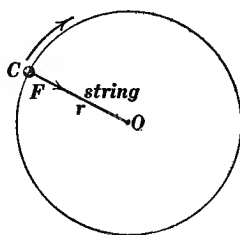


Fig. 78.

perform *translatory motion in a circle*. We may think of the entire mass of the stick as being concentrated at the center of mass, and therefore the stick in Fig. 77 may be thought of as equivalent to the very small ball  $C$ , in Fig. 78, the mass of the ball being the same as the mass of the stick. See page 44.

Let  $n$  be the number of revolutions per second of the point  $C$ , then the acceleration of  $C$  is at each instant towards  $O$  and equal to  $4\pi^2 n^2 r$  (or equal to  $v^2/r$ , where  $v$  is the velocity of  $C$ ), and  $4\pi^2 n^2 r m$  is the unbalanced force which must be pulling  $C$  towards  $O$ ,  $m$  being the mass of the stick.

**Example 1.**—A 5-pound ball is attached to a rotating shaft by means of a steel rod arranged like the spoke of a wheel. The center of the ball is 2 feet from the axis of the shaft and the

shaft makes 10 revolutions per second. Reducing the mass of the ball to slugs, we may use equation (1) of Art. 28 to calculate the force ( $4\pi^2 n^2 r m$ ) with which the rod must pull on the ball, namely, 1226 "pounds." It seems remarkable that so small a ball should require so large a force to hold it in its circular path, but such is the fact.

**Example 2.**—The center of mass of a circular saw is, let us say, 0.1 inch from the axis of the shaft on which the saw rotates, and we will assume that the shaft is held firmly in its bearings as it rotates so that the center of mass of the saw describes a circle of radius  $r = 1/120$  of a foot. The saw makes 3600 revolutions per minute ( $n = 60$  revolutions per second) and the mass of the saw is 10 pounds. What amount of

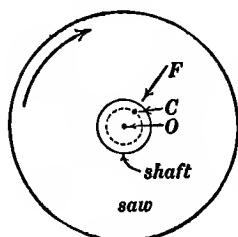


Fig. 79.

force must be exerted on the shaft to produce the necessary acceleration ( $4\pi^2 n^2 r$ ) of the center of mass of the saw? Reducing mass of saw to slugs we may use equation (1) of Art. 28 to calculate the force ( $4\pi^2 n^2 r m$ ), and it is 368 "pounds." This force is, of course, at each instant in the direction  $CO$  where  $C$  is the center of

mass of the saw and  $O$  is the axis of rotation, the force is indicated by the arrow  $F$ . See Fig. 79.

**41. Locomotive traveling on a railway curve.**—A locomotive not only moves forwards but it also rotates about a vertical axis when it rounds a curve. The rotatory motion of the locomotive may be understood if we think of the curve as a complete circle, then the locomotive will make one revolution about a vertical axis everytime it travels round the circle. If the velocity of the locomotive is constant in value the rotatory motion will be constant and only very small forces will have to be exerted on the locomotive because of its rotatory motion.\* Let us neglect these forces, then the only force or forces remaining to be considered are the forces which must act on the locomotive (a) to support it, and (b) to produce the translatory acceleration

\* Because of the asymmetry of the locomotive as explained in Art. 50.

$4\pi^2 n^2 r$  or  $v^2/r$  of the locomotive towards the center of circular path or curve along which the locomotive is traveling. These forces  $a$  and  $b$  are, of course, both exerted on the locomotive by the rails; therefore the rails must push upwards on the locomotive with a force  $mg$  to balance the downward pull of gravity on the locomotive, and the rails must exert a horizontal force equal to  $mv^2/r$  on the locomotive towards the center of the curve as indicated in Figs. 80 and 81. Therefore the total force exerted by the rails on the locomotive is the force  $R$ . If the track-face is at right-angles to  $R$  as indicated in Fig. 81, then the push of

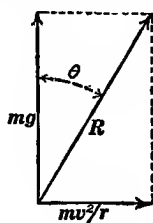


Fig. 80.

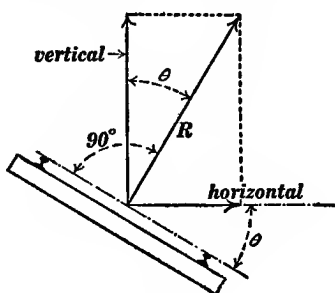


Fig. 81.

the rails on the locomotive wheels will be radial (rails will not push sidewise on wheel flanges). This condition is very much to be desired,\* and it is in fact realized if  $\tan \theta = (mv^2/r) \div (mg)$  or if  $\tan \theta = v^2/rg$ . This condition cannot always be met because the velocity  $v$  on a given curve is not always the same, but the outer rail on a railway curve is always elevated for the reason stated. This discussion applies to a lone locomotive. For a car in a long train the conditions are different because a train is under tension like a belt.

When a bicycle rider rounds a curve he must incline inwards and the angle of inclination,  $\theta$ , must be such that  $\tan \theta = v^2/rg$ . This is because the single† force  $R$ , Fig. 80, which is exerted

\* Let the reader consider why.

† Two forces are exerted on the bicycle by the ground, one on the front wheel and one on the back wheel, but these two forces would appear as a single force in a front or rear view of bicycle and rider.

on bicycle and rider by the ground must pass through the center of mass of bicycle and rider in order to balance the downward pull of gravity and produce the pure translational acceleration  $v^2/r$ —very small forces due to rotation of bicycle and rider about a vertical axis (see Art. 50), and small forces due to gyrostatic action of rotating bicycle wheels are here neglected.

**NOSING OF A LOCOMOTIVE.**—When a locomotive is on a straight track it does not rotate, and after the locomotive has entered a curve it does rotate about a vertical axis as above explained. Therefore *as the locomotive enters a curve* this rotatory motion must be suddenly established, and it is established by an excessive side-force exerted against the flanges of the front wheels of the locomotive by the outer rail. From the point of view of a man on the locomotive the flanges of the

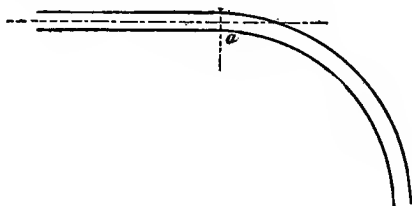


Fig. 82.

front wheels of the locomotive push with excessive force against the outer rail as the locomotive enters a curve. This action is called *nosing*. It is especially troublesome in the case of a locomotive with a short wheel base because it is the torque action of the side force on the front wheels that counts, and for a given side force this torque action is proportional to the length of wheel base. The electric locomotives of the New York, New Haven and Hartford Railroad, as first constructed, had very short wheel bases, and the nosing was troublesome until pilot trucks were placed at the ends.

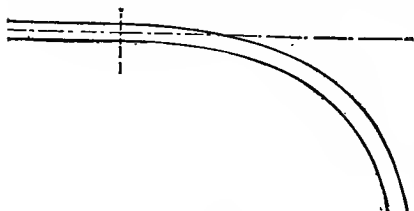


Fig. 83.

The electric locomotives of the New York, New Haven and Hartford Railroad, as first constructed, had very short wheel bases, and the nosing was troublesome until pilot trucks were placed at the ends.

Figure 82 shows a straight portion of track changing abruptly to a circular curve at the point *a*, and Fig. 83 shows the same straight portion changing gradually

into the same circular curve. The slow transition in Fig. 83 constitutes what is called an *easement curve*, and its object is to avoid the effects of abrupt entry of a locomotive into a curved portion of track as above described. The locomotive is set in rotation gradually as it traverses the easement curve.

**42. A belt on a rotating pulley.** Let  $T$  be the tension of a

belt  $m$  the mass of the belt per unit of length,  $O$  the outward push of pulley face on each unit length of belt,  $r$  the radius of the pulley, and  $v$  the velocity of the belt. Then

$$O = \frac{T}{r} - \frac{mv^2}{r} \quad (4)$$

When  $v = 0$  we have the case of the barrel hoop. See problem 55 on pages 36 and 37.

In the case of a ring which rotates like the rim of a wheel (without spokes) there is no outward push and  $O = 0$ .

In the case of a belt the push of unit length of the belt against the pulley face (which is equal and opposite to  $O$ ) grows less and less with increasing speed, and when  $mv^2/r$  is equal to  $T/r$  the belt does not push against the pulley face at all.

*Proof of equation (4).* Consider a very short portion  $ab$  of the belt on a rotating pulley as indicated by the black stretch  $ab$  in Fig. 84. Then  $T_1$  and  $T_2$  (each equal in value to the tension  $T$  of the belt) are the forces with which the adjacent portions of the belt pull on the portion  $ab$ , and  $T_1$  and  $T_2$  are at right angles to the respective radii  $r_1$  and  $r_2$ .

The resultant of  $T_1$  and  $T_2$  is the force which is represented by  $Ir \cdot \Delta\phi$  in Fig. 85, a force which is pulling inwards on the portion  $ab$  of the belt, and this

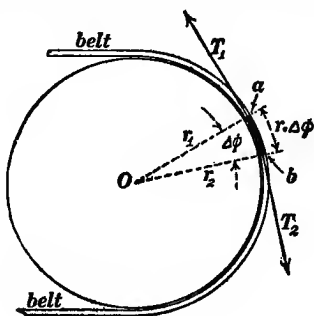


Fig. 84.

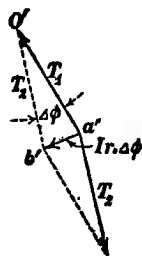


Fig. 85.

inward force on the portion  $ab$  is partly used to produce the radial acceleration  $v^2/r$  of  $ab$ , and partly used to balance the outward push of the pulley on the portion  $ab$ . Now the mass of  $ab$  is  $m \times r \cdot \Delta\phi$  so that the force required to produce the radial acceleration of  $ab$  is  $mr \cdot \Delta\phi \times v^2/r$ ; and, the outward push of the pulley face per unit length of belt being  $O$ , it is evident that the outward push on the portion  $ab$  whose length is  $r \cdot \Delta\phi$  is  $O \times r \cdot \Delta\phi$ . Therefore

$$Ir \cdot \Delta\phi = mv^2 \cdot \Delta\phi + Or \cdot \Delta\phi \quad (i)$$

But the triangle  $Oab$  in Fig. 84 is similar to the triangle  $O'a'b'$  in Fig. 85, so that

$$\frac{Ir \cdot \Delta\phi}{T} = \frac{r \cdot \Delta\phi}{r} \quad (\text{ii})$$

and if we substitute the value of  $Ir \cdot \Delta\phi$  from (ii) in (i) we get equation (4).

#### PROBLEMS.

**81.** A 200-gram ball is tied to a string and twirled in a circle of which the radius is 75 centimeters, and the ball makes 2 revolutions per second. (a) What is the velocity of the ball; (b) What is the acceleration of the ball; (c) What is the pull of the string on the ball?

**82a.** An 80-ton locomotive goes round a railway curve of which the radius is 600 feet at a velocity of 65 feet per second. With what horizontal force in "pounds" do the flanges of the locomotive wheels push against the outer rail, the outer rail not being elevated?

**82b.** Calculate the proper elevation to be given to the outer rail on a railway curve 600 feet in radius for a speed of 65 feet per second, the width of the track being 4 feet  $8\frac{1}{2}$  inches.

*Note.* Of course 600 feet radius is an unusually sharp railway curve. This problem refers strictly to a lone car or locomotive rounding the curve. A moving train is under tension like a belt, and its tension introduces a new condition.

**83.** A force of 978.1 dynes is required to support a one-gram body at the equator, but the one-gram body moves with the rotating earth making one revolution (actual) in about 23 hours 56 minutes, and the equatorial radius of the earth is about 6375 kilometers. Find acceleration of the apparently stationary one-gram body towards the center of the earth, and find the total actual gravity pull of the earth on the one-gram body in dynes.

**84.** Let us assume that the moon travels in a true circle around the earth under the influence of the earth's gravitational attraction. The moon's mean distance from the center of the earth is about 384,000 kilometers, and the time of one (actual) revolution of the moon around the earth is about 27 days  $7\frac{3}{4}$  hours. Find the acceleration of the moon towards the earth and find the force in dynes with which the earth would attract a one-gram body at the distance of the moon.

*Note.*—This problem was solved by Sir Isaac Newton when he wished to verify, his newly formed idea concerning the inverse square law of gravitation. The pull of the earth on a one-gram body at the equator is to the pull of the earth on a one-gram body at the distance of the moon as  $(384,000)^2$  is to  $(6375)^2$ . A very interesting discussion of Newton's law of gravitation is to be found in Young's *General Astronomy*, Ginn & Co., Boston, 1889; pages 108–116 and pages 256–282.

**85.** A horizontal arm ten feet long rotates about a vertical axis (like a spoke of a wheel). At the outer end of the arm, 10 feet from the axis, a swing is hung; and the length of the swing (end of arm to small heavy body in the swing) is 6 feet. Find the speed of the rotating arm which will make the swing stand out  $30^\circ$  from the vertical, taking the acceleration of gravity as 32 feet per second per second.

**86.** A car which is propelled by gravity has a velocity equal to  $\sqrt{2gd}$  wherever it may be when it is  $d$  feet below its starting point, if friction is negligible. Find how far above the top of a loop the starting point of a car must be to barely enable the car to loop the loop, the diameter of the loop being 20 feet.

**87a.** A water tank made of wood staves is 16 feet in diameter. What must be the tension of a steel hoop so that each foot of the hoop will push radially inward on the tank with a force of 200 “pounds”?

*Note.* This is, of course, a problem in statics, but it can be solved by using equation (4) of Art. 42. See problem 55 on pages 36 and 37.

**87b.** A belt has a mass of 0.6 pound per foot, its tension is 150 “pounds,” its velocity is 60 feet per second, and it runs on a pulley 4 feet in diameter. What is the inward push of each foot of belt on the pulley face?

*Note.*—It may be assumed in this problem that the belt is not driving the pulley, because if the belt does drive the pulley its tension is not the same everywhere.

**87c.** What is the critical speed of the belt in the previous problem, that is, the speed for which inward push of belt on pulley face is zero?

**87d.** A steel hoop 5 feet in diameter, total mass 150 pounds, rotates (like the rim of a wheel) at a speed of 1000 revolutions per minute. What is the tension in the hoop?

**87e.** The steel hoop of the previous problem can stand a tension of about 30,000 “pounds.” Find the highest speed it can stand.

43. **Hooke's law\* of elasticity and harmonic motion.**—Figure 86 shows a heavy ball  $B$  fastened to one end of a flat spring, the other end of the spring being clamped in a vise. The figure represents the ball  $B$  at rest and the spring unbent (straight). This position of the ball is called its equilibrium position. If the ball is pushed sidewise as indicated by the dotted lines the spring is bent and the force  $F$  with which the spring pushes the ball back towards the equilibrium position is:

$$F = - kx \quad (5)$$

where  $k$  is a constant which is called the *stiffness coefficient* of the spring. This equation expresses what is called Hooke's law of elasticity (distorting force proportional to distortion). If the spring is bent too far the proportional relationship fails.

If the ball in Fig. 86 is pushed to one side and released it will vibrate back and forth. This vibratory motion of the ball is of a kind that is very important in the theory of sound, and it is called *harmonic motion*. In the following discussion harmonic motion is defined ideally and the force action which is required to maintain it is determined by mathematical analysis. The force action thus determined is exactly in accordance with equation (5) and thus we infer that the ball  $B$  in Fig. 86 does in fact perform harmonic motion when it vibrates.

44. **Harmonic motion defined.**—A radius  $Op$ , Fig. 87, rotates steadily about the point  $O$  in the direction of the curved arrow, making  $n$  revolutions per second or  $2\pi n$  radians per second ( $= \omega$ ); and the particle  $P$  moves back and forth along the line  $LL$  so that the distance  $x$  is at all times given by the equation

$$x = r \cos \omega t$$

\* See Chapter VI for a more complete discussion of elasticity.

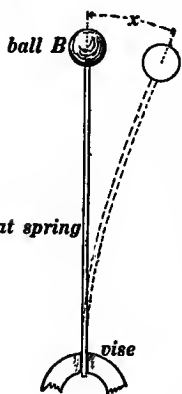


Fig. 86.

The motion of  $P$  so defined is called *harmonic motion*, and the number of round-trip vibrations of  $P$  per second (which is equal to  $n$  or to  $\omega/2\pi$ ) is called the *frequency* of the vibrations.

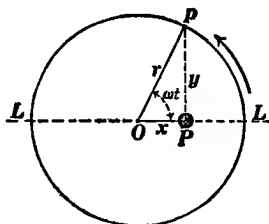


Fig. 87.

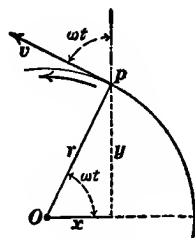


Fig. 88.

To determine the acceleration of  $P$  (and the unbalanced force which must act on  $P$ ) it is necessary to consider the differentiation of  $\cos \omega t$  and of  $\sin \omega t$ .

**45. Differentiation of cosine and sine.**—Let

$$x = r \cos \omega t \quad (i)$$

and

$$y = r \sin \omega t \quad (ii)$$

as indicated in Fig. 88. It is required to find expressions for  $\frac{dx}{dt}$

and for  $\frac{dy}{dt}$ , radius  $r$  being constant and  $\omega t$  being an angle which grows at the constant rate of  $\omega$  radians per second.

Let us consider the movement of the point  $p$  in Fig. 88 during an infinitely short interval of time  $\Delta t$ ; let  $\Delta x$  be its movement towards the right (it is actually moving towards the left so that  $\Delta x$  is negative), and let  $\Delta y$  be its movement upwards.

Then  $\frac{\Delta x}{\Delta t}$ , which is the  $x$ -component of the velocity of  $p$ , is the desired value of  $\frac{dx}{dt}$ ; and  $\frac{\Delta y}{\Delta t}$ , which is the  $y$ -component of the velocity of  $p$ , is the desired value of  $\frac{dy}{dt}$ ,

The velocity  $v$  of the point  $p$  is  $v = 2\pi nr$  or  $v = \omega r$ , ac-

according to second example in Art. 25, the  $x$ -component of  $v$  is  $-\omega r \sin \omega t$ , and the  $y$ -component of  $v$  is  $\omega r \cos \omega t$ . Therefore we have:

$$\frac{dx}{dt} = -\omega r \sin \omega t \quad (\text{iii})$$

and

$$\frac{dy}{dt} = +\omega r \cos \omega t \quad (\text{iv})$$

**46. Acceleration of  $P$  in Fig. 87 and unbalanced force which must act on  $P$ .**—The velocity of  $P$  is  $\frac{dx}{dt} = -\omega r \sin \omega t$ . Let us represent this velocity by the letter  $V$  and let us represent  $-\omega r$  (which is a constant) by the single letter  $a$ . Then

$$V = a \sin \omega t$$

and the acceleration of  $P$  is  $\frac{dV}{dt}$ . Looking at equations (ii) and (iv) of Art. 45 and thinking of  $y$ , for the moment, as standing for  $V$ , we see that  $\frac{dV}{dt} = \omega a \cos \omega t$ , whence, putting  $-\omega r$  for  $a$ , we get:

$$\frac{dV}{dt} = -\omega^2 r \cos \omega t$$

or, since  $\omega^2 = 4\pi^2 n^2$  and since  $x = r \cos \omega t$ , we have:

$$\left. \begin{array}{l} \text{acceleration of } P \\ \text{in Fig. 87} \end{array} \right\} = -4\pi^2 n^2 x$$

Multiplying this acceleration of  $P$  by the mass  $m$  of  $P$  we get an expression for the unbalanced force  $F$  which must act on  $P$  to cause  $P$  to move to and fro in the prescribed manner (so that  $x = r \cos \omega t$ ); therefore

$$F = -4\pi^2 n^2 m x \quad (6)$$

That is to say, the force  $F$  must be proportional to  $x$  (because in any given case the factor  $4\pi^2 n^2 m$  is a constant); but the force

acting on the ball in Fig. 86 is proportional to  $x$  according to equation (5). Therefore the ball in Fig. 86 performs harmonic motion and the factor  $k$  in equation (5) is

$$k = 4\pi^2 n^2 m \quad (7)$$

according to equation (6).

**Example.**—A force of 250 “pounds” pushes the ball in Fig. 86 0.1 foot to one side and therefore the stiffness coefficient of the spring is  $k = 2500$  “pounds” per foot. The mass of the ball is, let us say, 0.5 slug. Then from equation (7) we get  $n = 11.24$  complete vibrations per second. Of course c.g.s. units may be used throughout.

**47. The ideal pendulum** consists of a small ball  $B$ , Fig. 89, suspended by a rod or thread of negligible mass. Let  $l$  be the distance  $OB$  in Fig. 89 and let  $g$  be the acceleration of gravity. Then the number of complete vibrations of the pendulum per second will be

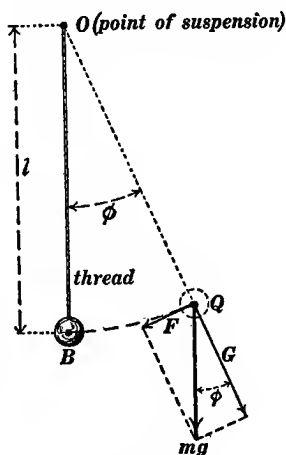


Fig. 89.

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \quad (8)$$

provided the pendulum swings through a small amplitude.

To establish equation (8) it is only necessary to get an expression for the “stiffness coefficient” (force  $F$ , Fig. 89, divided by distance  $BQ$ ) and place it equal to  $4\pi^2 n^2 m$  in accordance with equations (5), (6) and (7). Now the force  $F$  in Fig. 89 is  $mg \sin \phi$  which reduces to  $mg\phi$  if the angle  $\phi$  is small, and the distance  $BQ$  is  $l\phi$  so that equation (8) is easily obtained.

## PROBLEMS.

**88a.** An 8-pound body is performing harmonic oscillations and making 2 complete oscillations per second. What is its acceleration when it is 15 inches from its middle position (position of equilibrium)? What force is acting on it?

**88b.** The vibrating body of the previous problem reaches a maximum distance of 18 inches from the middle position. What is its velocity as it passes through the middle position? What is its kinetic energy? Does the spring store any potential energy at this instant? What is the potential energy of the spring when the body is at its maximum distance of 18 inches from the middle point?

**88c.** What is the kinetic energy of the body and what is the potential energy of the spring when the body in problem 88b is 9 inches from the middle position (maximum distance being 18 inches)?

**89.** A flat spring has one end fixed in a vise and a sidewise force of 5,000,000 dynes deflects the free end of the spring through a distance of 1.25 centimeters. What is the stiffness coefficient of the spring? A 2-kilogram ball is attached to the free end of the spring and set vibrating. How many complete vibrations per second does the ball make?

**90.** A 10-pound body is hung by a vertical helical spring and when set vibrating up and down it is observed to make 150 complete vibrations in two minutes. Find the "stiffness coefficient" of the spring. The stretch of the spring due to the weight  $mg$  of the suspended body is observed to be 6.187 inches. Find the acceleration of gravity, neglecting the mass of the spring.  
Ans. 31.80 feet per second per second.

*Note.*—The data of this problem are specified with extreme precision so as to justify accurate calculation. Divide mass in pounds by 32.174 to get slugs.

**91.** Find the length of a simple pendulum which will make one complete vibration per second at a place where the acceleration of gravity is 981 centimeters per second per second.

## CHAPTER III.

### SIMPLE DYNAMICS.

#### DYNAMICS OF ROTATION.

**48. Rotation about a fixed axis.**—The simplest kind of rotation is rotation about a stationary (fixed) axis, as exemplified by the rotation of a wheel mounted on a shaft.

**Angular velocity or spin velocity.**—Let  $\Delta\phi$  be the angle in radians turned by a wheel in  $\Delta t$  seconds. Then  $\Delta\phi/\Delta t$  is the *average angular velocity of the wheel* (in radians per second) during the time interval  $\Delta t$ , and the limiting value of  $\Delta\phi/\Delta t$  as  $\Delta t$  approaches zero is the *angular velocity of the wheel at a given instant*. In this text the angular velocity of a rotating body is called its *spin velocity*, it is represented by the letter  $s$ , and it is understood to be expressed in radians per second. Spin velocity is very often expressed in revolutions per second, and when so expressed it is represented by the letter  $n$ . One revolution is equal to  $2\pi$  radians, and therefore  $n$  revolutions per second is  $2\pi n$  radians per second, or  $s = 2\pi n$ .

**Angular acceleration or spin acceleration.**—Consider a wheel whose spin velocity is changing. Let  $\Delta s$  be the amount of spin velocity (radians per second) gained during  $\Delta t$  seconds, then  $\Delta s/\Delta t$  is the *average rate of gain of spin velocity (average spin acceleration)* of the wheel during the time  $\Delta t$ , and the limiting value of  $\Delta s/\Delta t$  as  $\Delta t$  approaches zero is the *spin acceleration of the wheel at a given instant*. Spin acceleration is always expressed in radians per second per second and it is represented by the letter  $\alpha$ . As an example consider an electric motor which is started from standstill and brought up to a speed of 20 revolutions per second in 6 seconds. The average spin acceleration of the motor armature during the 6 seconds is 20 revolutions per second divided by 6 seconds which gives 3.33 revolutions per second

per second, or 20.94 radians per second per second. Spin acceleration is understood to be expressed in radians per second per second throughout this chapter.

49. How does a wheel behave when an unbalanced torque acts on the wheel, the axis of the torque being coincident with the axis of the shaft on which the wheel is mounted?—The wheel gains spin velocity at a rate which is proportional to the torque, or the torque  $T$  required to produce a certain amount of spin acceleration  $\alpha$  is proportional to  $\alpha$  so that we may write

$$T = K\alpha \quad (9)$$

where  $K$  is a constant for a given wheel, and it is called the *moment of inertia* or *spin inertia* of the wheel.

50. Unbalanced torque must act on a steadily rotating nonsymmetrical body. Two balls each of mass  $m$  are carried by arms of length  $r$ , the arms being attached to a rotating axle as indicated in Fig. 90, and it is desired to find an expression for the unbalanced torque which must act on the axle at each instant when the axle, arms and balls rotate steadily  $n$  revolutions per second, mass of arms neglected. A force  $F(= 4\pi^2 n^2 m r)$  must

act as indicated in the figure, and a force  $G(= 4\pi^2 n^2 m r)$  must act as indicated; and these two forces together constitute a torque  $T = 4\pi^2 n^2 m r L$  about an axis which is at each instant at right angles to the plane which contains the axle and the two arms  $r$ . The two balls  $mm$  in Fig. 90 have a certain kind of asymmetry with respect to the axle, and any body which has this kind of asymmetry with respect to an axis of rotation cannot rotate steadily about that axis unless acted upon at each instant by an unbalanced torque.

CONSERVATION OF SPIN MOMENTUM.—The product  $Ks$  is called the *angular momentum* or *spin momentum* of a rotating body,  $s$  being its spin velocity and

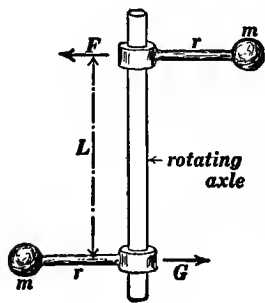


Fig. 90.

$K$  its spin inertia. The spin momentum of a single rotating body (symmetrical or non-symmetrical) is constant when the body is not acted on by an unbalanced torque and the combined spin momentum of a number of bodies which act on each other is not changed by mutual force actions between the bodies. See discussion of conservation of translatory momentum in Art. 38. A very striking experiment illustrating the principle of the conservation of spin momentum is described in Art. 70. The principle of the conservation of spin momentum is of very great importance in the theory of the free rotation of a non-symmetrical body. The most intelligible discussion of this problem is Poisson's *Theorie Nouvelle de la Rotation des Corps*.

**51. The moment of inertia or spin inertia of a given body varies with the position of the axis of rotation.** Experiment with a slim stick.—A slim stick may be very easily set spinning about the axis  $OO$  in Fig. 91, less easily about the axis  $OO$  in Fig. 92, and still less easily about the axis  $OO$  in Fig. 93. That is to



Fig. 91.

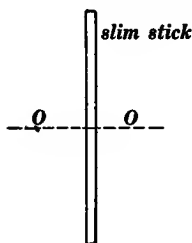


Fig. 92.

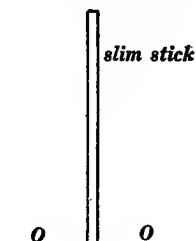


Fig. 93.

say, a given torque would produce a given spin velocity in a very short time about the axis  $OO$  in Fig. 91, the same torque would have to act much longer to produce the same spin velocity about the axis  $OO$  in Fig. 92, and it would have to act still longer to produce the same spin velocity about the axis  $OO$  in Fig. 93. These statements are easily verified by trial. Therefore the spin-inertia of the stick is small about the axis  $OO$  in Fig. 91, it is larger about the axis  $OO$  in Fig. 92, and still larger about the axis  $OO$  in Fig. 93.

**52. Contribution to spin inertia by a small particle (of a body) of mass  $\Delta m$  at a distance  $r$  from the axis of spin.**—Consider a wheel upon which a torque  $T$  acts and produces spin acceler-

ation  $\alpha$  about the axis  $O$  as indicated in Fig. 94. A certain portion  $\Delta T$  of the torque is used to speed up the particle  $\Delta m$  or, in other words, a certain portion  $\Delta K$  of the spin inertia of the wheel is due to  $\Delta m$ , so that, according to equation (9) of Art. 49, we have

$$\Delta T = \Delta K \cdot \alpha \quad (i)$$

It is required to find an expression for  $\Delta K$ .

The sidewise velocity of the particle  $\Delta m$  (velocity in the direction of  $\Delta F$ ) is  $v = rs$ , where  $s$  is the spin velocity of the wheel. This is evident when we consider that  $s \cdot \Delta t$  is the angle in radians turned by the radius  $r$  in time  $\Delta t$  so that  $r \times s \cdot \Delta t$  is the length of the very small arc over which  $\Delta m$  travels during the time  $\Delta t$ , and the velocity of  $\Delta m$  is the distance traveled divided by  $\Delta t$ .

Consider any quantity  $y$  which is always  $b$  times as large as  $x$ , then evidently  $y$  must always change  $b$  times as fast

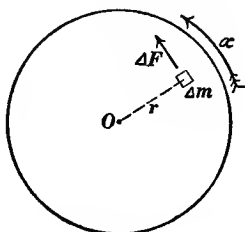


Fig. 94.

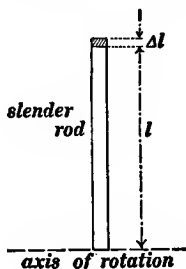


Fig. 95.

as  $x$ . Now  $v$  is  $r$  times  $s$ , therefore  $v$  must change  $r$  times as fast as  $s$ ; but the rate of change of  $v$  is the sidewise acceleration of  $\Delta m$ , and the rate of change of  $s$  is  $\alpha$ . Therefore the sidewise acceleration of  $\Delta m$  is equal to  $r\alpha$ .

Let the force which sets  $\Delta m$  in motion be  $\Delta F$ , then, according to equation (1) of Art. 28,  $\Delta F = \Delta m \times r\alpha$ . But the torque action of  $\Delta F$  about  $O$  is  $r \cdot \Delta F$  which is  $\Delta T$  in equation (i). Therefore  $\Delta T = r^2 \cdot \Delta m \times \alpha$  so that

$$\Delta K = r^2 \Delta m \quad (10)$$

*Therefore if we multiply the mass of each particle of a wheel by the square of its distance from the axis of rotation and add all such products together we will get the spin inertia of the wheel.*

**53. Calculus discussion of the spin inertia (moment of inertia) of the slender stick in Fig. 93.**—Imagine the stick to grow longer by the infinitely small amount  $\Delta l$  as indicated in Fig. 95. What is the increase of  $K$ ? Let  $\mu$  be the mass of unit length of the stick. Then  $\mu \Delta l$  is the added mass which is represented by the shaded area in Fig. 95, and  $l^2 \mu \Delta l$  is the increase of  $K$  according to equation (10) of Art. 52. Therefore

$$\Delta K = \mu l^2 \Delta l$$

from which we get

$$\frac{dK}{dl} = \mu l^2 \quad (i)$$

and this equation expresses the law of growth of  $K$  when the stick is imagined to grow in length. It is required to find an expression for  $K$ , and the method to be used is the method of integration as explained in Art. 27.

*Finding of a function which has the same derivative as  $K$ .*—

Consider the function  $y = bl^3$  which gives  $\frac{dy}{dl} = 3bl^2$ . Evidently if we make  $3b = \mu$  or  $b = \mu/3$  we will have:

$$y = \frac{1}{3}\mu l^3 \quad \text{and} \quad \frac{dy}{dl} = \mu l^2$$

so that  $y$  is a function which has the same derivative as  $K$ .

*Finding of expression for  $K$ .*—Therefore  $K$  must be equal to  $y + C$  or  $\frac{1}{3}\mu l^3 + C$ , where  $C$  is an undetermined constant.

*Determination of constant of integration.* When  $l$  is zero it is evident that  $K$  is zero. Therefore  $C$  must be zero, and the correct expression for  $K$  is  $K = \frac{1}{3}\mu l^3$ . This expressions may be somewhat simplified by writing  $m$  (the mass of the stick) for  $\mu l$  which gives

$$K = \frac{1}{3}ml^2$$

The following table gives the values of the spin inertia of several regular homogeneous solids.

TABLE.

Values of Spin-Inertia of Some Regular Homogeneous Solids.

Shape of solid and position of axis.	Value of $K$ .
Solid sphere of radius $r$ and mass $m$ , <i>axis of rotation passing through center of sphere</i> .....	$\frac{2}{5}mr^2$
Solid cylinder of radius $r$ and mass $m$ , <i>axis of rotation same as axis of cylinder</i> .....	$\frac{1}{2}mr^2$
Very slim rod of length $l$ and mass $m$ , <i>axis of rotation at right angles to rod and passing through center of rod</i> .....	$\frac{1}{12}ml^2$
Rectangular parallelepiped of length $l$ and breadth $b$ and mass $m$ , <i>axis of rotation at right angles to <math>l</math> and <math>b</math>, and passing through center of parallelepiped</i> .....	$\frac{1}{12}m(l^2 + b^2)$

**Note.** The axis of rotation is understood to pass through the center of mass of the body as indicated by statements in Italics in the accompanying table. Let  $K$  be the spin inertia of a body about an axis passing through the center of mass of the body, and let  $K'$  be the spin inertia about another axis *parallel to the first and distant  $a$  from it*. Then

$$K' = K + a^2m \quad (11)*$$

where  $m$  is the mass of the body.

**Units of spin inertia.**—The spin inertia of a body is, according to the above discussion, expressed as the product of a mass times the square of a distance or length. The c.g.s. unit of spin inertia, is one *gram-(centimeter)<sup>2</sup>*, the f.s.s. unit of spin inertia is one *slug-(foot)<sup>2</sup>*. One gram at a distance of one centimeter from the axis of rotation represents one c.g.s. unit of spin inertia, and one slug at a distance of one foot from the axis of rotation represents one f.s.s. unit of spin inertia.

**54. Radius of gyration.**—The spin inertia of a cylinder with respect to the axis of figure of the cylinder is  $\frac{1}{2}mr^2$  according to the table in Art. 53. Imagine the entire mass  $m$  of the cylinder

\* A proof of this equation is given in Franklin and MacNutt's *Mechanics and Heat*.

to be concentrated at a point at a distance  $\rho$  from the axis. Then the spin inertia of the cylinder would be  $\rho^2 m$  according to equation (10). What value must the distance  $\rho$  have in order that  $\rho^2 m$  may be equal to the spin inertia of the cylinder  $\frac{1}{2}mr^2$ ? Placing  $\rho^2 m = \frac{1}{2}mr^2$  we get  $\rho = r/\sqrt{2}$ . The distance  $\rho$  so defined is called the *radius of gyration* of the cylinder. The radius of gyration of any body *with respect to a given axis* is defined by the equation  $m\rho^2 = K$  where  $m$  is the mass of the body and  $K$  is its spin inertia with respect to the given axis.

## PROBLEMS.

92. A very slender rod 110 centimeters long has a mass of 550 grams. Calculate its spin inertia about an axis passing through its middle and at right angles to it. Calculate  $K$  by using the formula which is given in the table in Art. 53, and calculate  $K$  approximately by multiplying the mass of each 10-centimeter section of the rod by the square of the distance of the middle point of the section from the middle point of the rod—all such products being added.

93. A wheel and axle has a spin inertia of 25 slug(-feet).<sup>2</sup> The wheel is brought up to a speed of 1200 revolutions per minute and then left to itself, and a revolution counter indicates 3700 revolutions while the wheel is coming to rest. Assuming wheel to lose spin velocity at a constant rate, find (a) Time in seconds to come to rest, (b) Average rate of loss of spin velocity while stopping, and (c) Average value of retarding torque which stops the wheel and axle.

*Note.*—The argument of this problem is exactly similar to the argument of example 1 in Art. 35.

94. Let  $K$  be the spin inertia of a disk of radius  $r$  (referred to axis of figure of disk). Imagine the disk to grow by the addition of a ring of material so as to increase its radius from  $r$  to  $r + \Delta r$ , where  $\Delta r$  is infinitely small. Find an expression for  $\Delta K$ , and thus find an expression for  $\frac{dK}{dr}$ . In this development

let  $\mu$  be the mass of the disk per unit of area. Differentiate the function  $y = ar^4$  and find a function which has the same derivative as  $K$ . Then find the correct expression for  $K$  in terms of  $\mu$  and  $r$ . Substitute  $m$  for  $\pi r^2 \mu$  in this expression for  $K$  and get  $K$  in terms of  $m$  and  $r$ .

95. Find the radius of gyration of a very slim rod 100 centimeters long with respect to an axis passing through the middle of the rod and at right angles to the rod.

96. Find the radius of gyration of the slim rod of the previous problem with respect to an axis passing through the end of the rod and at right angles to the rod.

97. A homogeneous ball 20 centimeters in diameter is fixed to a rotating shaft so that the center of the ball is 30 centimeters from the axis of the shaft. The mass of the ball is 65 kilograms. What is the spin inertia of the ball with respect to the axis of the shaft and what is the radius of gyration?

**55. Example of combined translation and rotation.**—A homogeneous cylinder of mass  $m$  and radius  $r$  rolls down a  $\theta^\circ$  incline, starting from rest. How far does the cylinder roll in  $t$  seconds, the acceleration of gravity being  $g$ ?

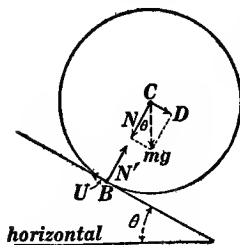


Fig. 96.

Only two forces act on the cylinder (friction of air being neglected), namely, the downward pull of gravity  $mg$  and the force  $F$  exerted on the cylinder by the inclined plane as indicated in Fig. 96. The downward force  $mg$  may be replaced by its two components, namely, the down-hill component  $D = mg \sin \theta$  and the normal-to-plane component  $N = mg \cos \theta$ . Also

the force  $F$  may be replaced by its two components, namely, the up-hill component  $U$  and the normal-to-plane component  $N'$ . We may then consider the four forces  $D$ ,  $N$ ,  $U$  and  $N'$ .

(a) The cylinder has no normal-to-plane acceleration. Hence forces  $N$  and  $N'$  balance, or

$$N' = mg \cos \theta \quad (i)$$

(b) The cylinder has an unknown translatory acceleration  $a$  down-hill. Therefore the net down-hill force,  $D - U$ , must be equal to  $ma$ . That is

$$D - U = ma \quad (ii)$$

(c) The cylinder rotates faster and faster as it travels down hill; that is to say, the cylinder has an unknown spin acceleration  $\alpha$  about an axis passing through  $C^*$ . The force  $U$  is the only force which has a torque action about  $C$  and its torque action about  $C$  is  $T = Ur$ . Therefore, according to equation (9) of Art. 49, we have:

$$Ur = K\alpha \quad (iii)$$

where  $K$  is the spin inertia of the cylinder about an axis through  $C$  (see table in Art. 53).

(d) While the cylinder makes one revolution it travels a distance  $2\pi r$ , and if it makes  $n$  revolutions per second it travels  $2\pi nr$  feet per second (or centimeters per second). Therefore the velocity of the cylinder is  $v = 2\pi nr = sr$ , where  $s$  is the spin velocity of the cylinder, and  $r$ , the radius of the cylinder, is a constant. If  $v = rs$  it is evident that the rate of change of  $v$  (which is  $a$ ) must be  $r$  times the rate of change of  $s$  (which is  $\alpha$ ).<sup>\*</sup> Therefore:

$$a = r\alpha \quad (iv)$$

<sup>\*</sup>It is permissible to think of  $\alpha$  as being about any axis at right angles to the plane of the paper in Fig. 96; but the problem is greatly simplified by thinking of  $\alpha$  as taking place about an axis through  $C$ . To show that this is permissible it is only necessary to show that the two forces  $D$  and  $U$  are together equivalent to a down-hill force  $D - U$  acting at  $C$  and producing the translatory acceleration  $a$ , and a pure torque  $Ur$  which produces, the spin acceleration  $\alpha$  about  $C$ . Let the reader work the matter out for himself.

† If John always has 10 times as much money as Henry he must always save money 10 times as fast as Henry.

From the four equations (i), (ii), (iii) and (iv) the values of  $N'$ ,  $U$ ,  $a$  and  $\alpha$  may be calculated;  $m$ ,  $g$ ,  $\theta$ ,  $K$  and  $r$  being known.\* From the value of  $a$  so found (a constant acceleration)

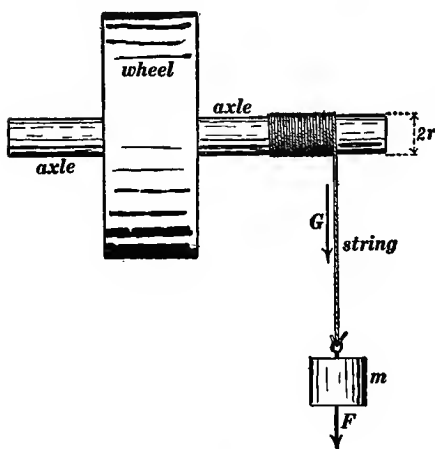


Fig. 97.

the distance traveled by the cylinder in  $t$  seconds can be calculated as explained in Art. 34.

56. Another example of combined translation and rotation.—Figure 97 shows a wheel and axle being set in rotation by the pull of a string on which a body of mass  $m$  is hung. Given the mass  $m$ , the acceleration of gravity  $g$ , the radius of the axle  $r$  and the distance

$d$  fallen by  $m$  in  $t$  seconds (starting from rest) to calculate the spin inertia of the wheel.

(a) Let  $G$  be the tension of the string. Then the forces acting on  $m$  are a downward force  $mg$  and an upward force  $G$ , therefore the net downward force is  $mg - G$ , so that

$$mg - G = ma \quad (i)$$

where  $a$  is the downward acceleration of  $m$ .

(b) The torque acting on the wheel (axle friction being assumed equal to zero) is  $Gr$ , so that

$$Gr = K\alpha \quad (ii)$$

(c) As in the previous example we have

$$a = r\alpha \quad (iii)$$

(d) From the known distance  $d$  traveled downwards by  $m$

\* In fact  $r$  cancels out and need not be known.

in a known time  $t$  we can calculate the downward acceleration  $a$  as explained in Art. 34. Then knowing  $a$ ,  $r$ ,  $m$  and  $g$  we can calculate  $G$ ,  $K$  and  $\alpha$  by means of equations (i), (ii) and (iii).

**57. Harmonic rotatory motion.**—To every equation relating to translatory motion there is a corresponding equation, identical in form, relating to rotatory motion.\* Thus  $F = ma$  [equation (1) of Art. 28] corresponds to  $T = K\alpha$  [equation (9) of Art. 49]. It is the purpose of this article to point out equations in rotatory motion which correspond to  $F = -kx$  [equation (5) of Art. 43] and to  $k = 4\pi^2 n^2 m$  [equation (7) of Art. 46].

A body is suspended by a steel wire. When the body is turned through the angle  $\phi$  the wire is twisted, and the twisted wire exerts a torque  $T$  on the body such that

$$T = -k'\phi \quad (12)$$

where  $k'$  is a constant which is called the *coefficient of torsional stiffness of the wire*.

If the suspended body is turned and released it will oscillate back and forth like the balance wheel of a watch, performing harmonic rotatory motion, and:

$$4\pi^2 n^2 K = k' \quad (13)$$

where  $n$  is the number of complete vibrations per second and  $K$  is the spin inertia of the suspended body with respect to the suspending wire as an axis. The angle  $\phi$  in equation (12) must be expressed in radians.

**Example.**—A 1000-gram disk of 10 centimeters radius [ $K = 50,000$  gram-(centimeters)<sup>2</sup>] makes 53 complete vibrations in 360 seconds when hung by a steel wire, axis of figure of disk being coincident with wire. How much torque would be required to twist the free end of the wire through two radians of angle? The value of  $k'$  as calculated by equation (12) is 42,800 dyne-

\* The converse of this statement is not true by any means. This matter is fully set forth in Appendix C.

centimeters of torque per radian of twist, and if the angle of twist is two radians the torque must be 85,600 dyne-centimeters.

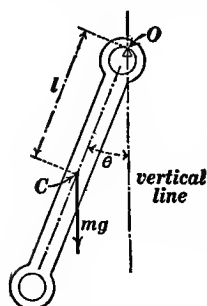


Fig. 98.

**58. The physical pendulum.**—Figure 98 represents a bar swinging freely as a pendulum about an axis  $O$ ; the center of mass (center of gravity) of the bar is at  $C$ . The torque action about  $O$  of the force  $mg$  is  $mg l \sin \theta$ , or  $mg l \theta$  if  $\theta$  is small. Therefore the stiffness coefficient is  $mg l$  and this must be equal to  $4\pi^2 n^2 K'$  according to equation (13), so that

$$4\pi^2 n^2 K' = mg l \quad (i)$$

where  $n$  is the number of complete vibrations per second and  $K'$  is the spin inertia of the bar about an axis through  $O$ .

#### PROBLEMS.

**98.** Find the acceleration of a solid homogeneous sphere rolling down a  $30^\circ$  incline, acceleration of gravity being 980 centimeters per second per second.

**99.** A 10,000-gram disk is 20 centimeters in diameter, it is mounted on a shaft 2 centimeters in diameter of the same material and the projecting parts of the shaft have a mass of 100 grams. The disk rolls 109.2 centimeters down an inclined track in 15 seconds, the drop of the track being one twentieth of its length along its slope. What is the acceleration of gravity?

*Note.*—The total mass of the rolling body is 10,100 grams, and its total spin-inertia is the sum of the spin-inertia of the complete disk and the spin-inertia of the projecting part of the axle.

**100.** The disk and axle of problem 99 has two very fine wires wrapped around the ends of the axle, and, holding these wires fast, the disk is allowed to fall and set itself spinning as the wires unroll from the axle. Find downward acceleration of disk and

find combined upward pull of the two wires, acceleration of gravity being 980 centimeters per second per second.

*Note.*—The diagram which is shown in Fig. 99 will suggest the formulation of this problem.

101. A forward pull  $P$  is exerted on a 500-pound wheel which rolls on a level floor. The diameter of the wheel is 6 feet and

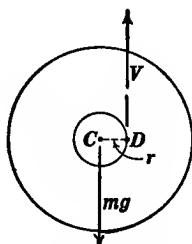


Fig. 99 (Old 99)

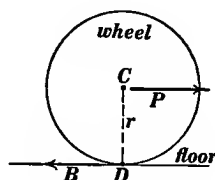


Fig. 100 (Old 100)

its spin inertia about an axis passing through its center is 1920 pound-(feet)<sup>2</sup>. Find the forward acceleration of the wheel when  $P = 10$  "pounds":—(a) When  $P$  is exerted at the center of the wheel, and (b) When  $P$  is exerted at the top of the wheel.

*Note.* The formulation of this problem will be suggested by Fig. 100

102. Solve problem 101 by considering the energy relations.

*Note.* Let the wheel start from rest, let  $x$  be the distance traveled in  $t$  seconds, and let  $a$  be the constant acceleration. Then in case (a), when  $P$  acts at the center of the wheel, we have: Work done  $= Px = \frac{1}{2}mv^2 + \frac{1}{2}Ks^2$ ; also we have  $v = rs$ ,  $v = at$  and  $x = \frac{1}{2}at^2$ . From these equations  $a$  can be calculated.

103. A 500-gram disk, 20 centimeters in diameter, is suspended by a steel wire and set oscillating, like the balance wheel of a watch, about the steel wire as an axis. It is found by observation that the disk makes 11 complete vibrations in 249 seconds. Find the coefficient of torsional stiffness of the steel wire.

104. A steam-engine connecting rod is swung as a pendulum as indicated in Fig. 98. The mass of the rod is 15 pounds (to be

reduced to slugs, of course), the distance  $l$  is 2.42 feet, the local acceleration of gravity is 32.2 feet per second per second, and the pendulum is observed to make 51 complete vibrations in 120.2 seconds. Calculate the spin inertia of the connecting rod about an axis perpendicular to the plane of the paper and passing through the point  $C$  in Fig. 98. Ans. 2.37 slug-(feet).<sup>2</sup>

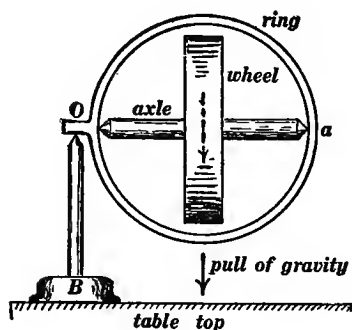


Fig. 101.

59. Rotation about a rotating axis. The gyroscope.—Figure 101 shows a rapidly spinning wheel mounted in a ring which rests on a pointed post  $BO$ . The pull of gravity on ring and wheel produces an unbalanced torque action about an axis through  $O$  (axis of torque horizontal and at right angles to axis of spinning wheel), and this

unbalanced torque causes the wheel and ring to swing round and round the post, the axis  $Oa$  retaining its horizontal position.\* This arrangement is called a *gyroscope*. A complete discussion† of the gyroscope is, of course, beyond the scope of this text, and therefore the following purely geometric discussion must suffice.

*Representation of a torque by an arrow.*—Figure 102 represents a torque acting on a screw driver. To represent this torque by an arrow draw the arrow  $T$  in the direction of the axis of the torque, make the length of the arrow represent the value of the torque in dyne-centimeters or in "pound"-feet to any convenient scale, and place the arrow-head to indicate the direction in which a right-handed screw would travel if turned by the torque.

*Representation of spin velocity by an arrow.*—Figure 103 repre-

\* Except for effect due to friction.

† See references on introductory pages.

sents a spinning wheel. To represent the spin velocity  $s$  of the wheel by an arrow draw the arrow  $s$  in the direction of the axis of spin, place the arrow head so as to indicate the direction

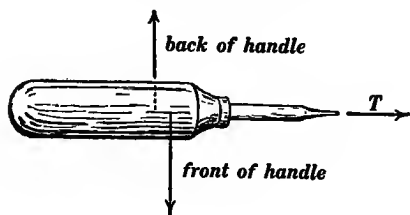


Fig. 102.

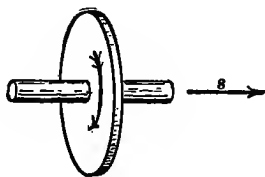


Fig. 103.

in which the spinning wheel would travel if it were a nut turning on a right-handed screw, and make the length of the arrow represent the value of the spin velocity in radians per second to any convenient scale.

*The importance of representing torques and spin velocities by arrows is due to the fact that both torques and spin velocities are added by the parallelogram law.\**

**Vector diagram of gyroscopic motion.**—Figure 104 represents the gyroscope of Fig. 101 as seen from above at a given instant.

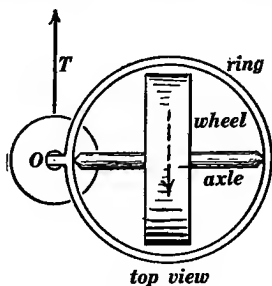


Fig. 104.

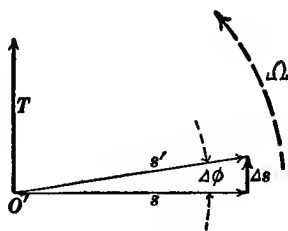


Fig. 105.

At this instant the spin velocity of the wheel is represented by the arrow  $s$  in Fig. 105, and the torque due to gravity pull on ring and wheel is represented by the arrow  $T$  in Fig. 105. During a short interval of time  $\Delta t$  the torque  $T$  produces a

\* See Franklin and MacNutt's *Mechanics and Heat*, pages 176-177.

certain amount of spin velocity  $\Delta s$ , and this spin velocity added to  $s$  gives  $s'$  which is the axis of the spinning wheel after lapse of time  $\Delta t$ .

The rate at which the torque  $T$  produces spin velocity is  $\alpha = T/K$  according to equation (9) of Art. 49, and therefore  $\Delta s = \alpha \cdot \Delta t = (T/K) \cdot \Delta t$ . But the angle between  $s$  and  $s'$  in radians is equal to  $\Delta s/s$  or to  $(T/K) \cdot \Delta t/s$ , and if we divide this angle in radians by  $\Delta t$  we get the rate at which the axis of spin swings round  $O$ . Representing this rate of swinging of axis of spin round  $O$  by the Greek letter  $\Omega$ , we get  $\frac{T}{K} \times \frac{1}{s} = \Omega$ , or

$$T = Ks\Omega \quad (14)$$

**Example 1.**—The spin inertia  $K$  of the paddle wheel of a side-wheel boat is 12,500 slug-(feet),<sup>2</sup> the spin velocity  $s$  of the paddle wheel is 6 radians per second, and the boat is turning to port so as to describe a complete circle in  $20\pi$  seconds, so that  $\Omega = 0.1$  radian per second. A torque  $T$  equal to  $Ks\Omega$  or 7500 “pound”-feet must act on the paddle wheel shaft; and to prevent the boat from listing a 750-“pound” weight would have to be shifted from the center line of the boat to a point 10 feet to one side of the center line. To which side would the weight have to be shifted?

**Example 2.**—Figure 106 is a top view of an automobile turning to the right. The spin velocity of the engine fly wheel at a given instant is represented by the arrow  $S$  in Fig. 107, and  $\Delta t$  seconds later the spin velocity of the fly wheel is as represented by the arrow  $S'$ . The increment of spin velocity is represented by the arrow  $\Delta S$ , and the torque which must act on the fly-wheel shaft is represented by the arrow  $T$ . Therefore while the car is turning to the right the rear bearing must push upwards on the fly-wheel shaft and the front bearing must pull downwards on the fly-wheel shaft. These forces are, of course, in addition to the forces which must be exerted on the fly-wheel shaft to support wheel and shaft against the downward pull of gravity.

Let  $r$  be the radius of the curve described by the automobile and  $v$  the velocity of the automobile. Then thinking of the car as going round a complete circle it is easy to see that  $\Omega =$

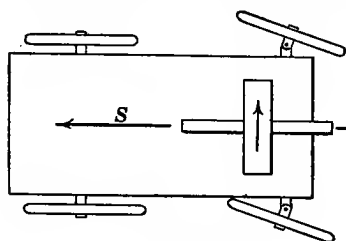


Fig. 106.

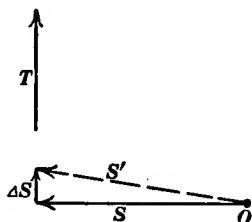


Fig. 107.

$v/r$  so that the torque  $T$  in Fig. 107 would be  $T = Ksv/r$  according to equation (14).

#### PROBLEMS.

**105.** Make a diagram somewhat similar to Figs. 106 and 107 for the case in which the front wheels rise suddenly in going over a bump on the road, and state precisely what forces are exerted by the fly wheel shaft on the bearings (gyrostatic reaction forces).

**106.** The entire ring and wheel in Fig. 101 has a mass of 150 grams, the center of mass (center of gravity) of wheel and ring is 6 centimeters from the point  $O$  and the acceleration of gravity is 980 centimeters per second per second. Calculate the torque in dyne-centimeters which is represented by the arrow  $T$  in Fig. 105.

The spin inertia of the wheel is 900 gram-(centimeters)<sup>2</sup> and the wheel makes, let us say, 4500 revolutions per minute. Calculate the time required for the wheel and ring to swing once around the vertical post  $BO$  in Fig. 101.

**107.** A side-wheel boat is steered in a circle of 150 feet radius at a velocity of 25 feet per second, and the boat lists  $5^\circ$  because of gyrostatic action of paddle wheels and shaft. To produce a  $5^\circ$  list when the boat is standing still requires a weight of 20,000 "pounds" to be shifted 15 feet to one side from the center line of the boat. The paddle wheels make 75 revolutions per minute. Find the spin inertia of paddle wheels and shaft.

**108.** The spin inertia of the engine shaft and flywheel of a Ford automobile is about 30 pound-(feet)<sup>2</sup>. The engine speed is, say, 600 revolutions per minute, and the car turns in a circle of 25-foot radius at a speed of 30 miles per hour. Find the torque which must act upon the flywheel shaft to make the flywheel and shaft turn round with the car. Suppose the distance between front and back bearings is 2 feet, and suppose that the above mentioned torque is produced by equal and opposite forces at the bearings. Find the value of each force and specify the direction of each force when the car turns to the right.

*Note.*—The automobile is here assumed to turn an excessively sharp curve so as to make the gyroscopic forces appreciable. Ordinarily these forces are entirely negligible.

**109.** The armature shaft of a ship's dynamo is horizontal and at right angles to the ship's keel. Make a diagram showing an assumed direction of rotation of the armature and showing forces exerted on bearings by armature shaft (gyroscopic reaction of spinning armature) while the port side of the ship is rising. The maximum velocity of roll of the ship is, let us say, 10° per second. Calculate value of each force above mentioned, spin inertia of armature being 150 slug-(feet)<sup>2</sup>, speed of armature being 1500 revolutions per minute, and distance between armature bearings being 4 feet.

## CHAPTER IV

### HYDROSTATICS.

**60. Hydrostatic pressure.**—*A fluid\* at rest always pushes normally against a surface which is exposed to the action of the fluid.* Thus the small arrows in Fig. 108 show how the steam in a steam-engine cylinder pushes against the piston, the arrows in Fig. 109 show how the water in a pail pushes against the sides

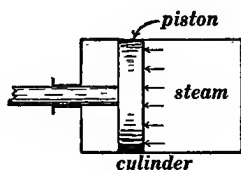


Fig. 108.



Fig. 109.

of the pail (of course the water pushes on the bottom of the pail also), and the arrows in Fig. 110 show how the water in a tank pushes against the surface of a submerged ball.

*A fluid at rest not only pushes on a surface which is exposed to its action, but two contiguous portions of the fluid push on each other as shown in Fig. 111.* The line *qq* represents a small plane area or surface which may be imagined to separate two contiguous portions of the fluid, and the small arrows show how the portions of the fluid push on each other.

*A fluid in motion does not necessarily push normally against a surface which is exposed to the action of the fluid.* Thus the small arrows in Fig. 112 show how the moving water in a pipe pushes against the walls of the pipe. Also, when a fluid is moving the forces which are indicated by the arrows in Fig. 111 are not necessarily at right angles to the plane area *qq*. Thus the small arrows in Fig. 113 show how the portions of the moving

\* The term *fluid* includes both liquids and gases.

water on the two sides of the plane  $qq$  push on each other

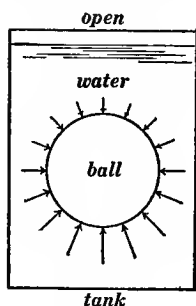


Fig. 110.



Fig. 111.

across  $qq$ . The remainder of this chapter deals with fluids at rest.

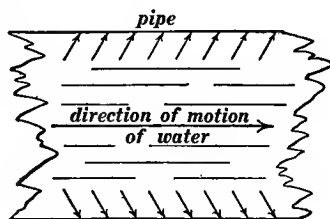


Fig. 112

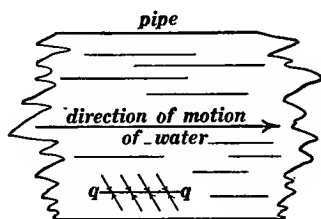


Fig. 113.

**Pascal's Principle.**—The force  $\Delta F$  with which a fluid at rest pushes against a small portion  $\Delta a$  of an exposed surface is the same in value whatever the direction of  $\Delta a$ . This proposition is known as *Pascal's principle* from its discoverer. It can be proven theoretically from the proposition that  $\Delta F$  is always at right angles to  $\Delta a$ , but a simple experimental verification is more satisfactory here. Fig. 114

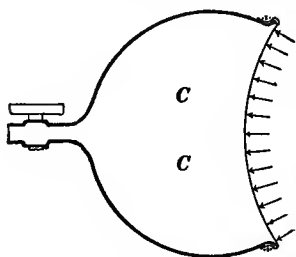


Fig. 114.

shows a glass cup  $C$  over the mouth of which a thin sheet of rubber is tied. The air is partly drawn out of the cup so that the rubber sheet is pushed inwards by

the outside air. Then, with the stop-cock closed, the dished shape of the rubber sheet remains unchanged as the cup is turned so as to make the rubber sheet face upwards or downwards or sidewise.

**Definition of hydrostatic pressure.**—Let  $\Delta F$  be the force exerted by a fluid at rest on a small portion  $\Delta a$  of an exposed surface,  $\Delta a$  being expressed, of course, in square centimeters or in square feet. *The quotient  $\Delta F/\Delta a$  approaches a definite limiting value  $p$  if  $\Delta a$  is taken smaller and smaller*, and the limiting value  $p$  is called the *hydrostatic pressure* or simply the *pressure* of the fluid at the point.\* Therefore we have:

$$\Delta F = p \cdot \Delta a \quad (15)$$

The above statement may be unintelligible to the student who is not thoroughly familiar with the idea of limits, and the following statement may therefore be taken as the definition of the pressure of a fluid: The force  $F$  which is exerted by a fluid at rest on an exposed flat surface  $a$  is *proportional to  $a$*  and the quotient  $F/a$  is called the pressure of the fluid. That is

$$F = pa$$

This equation is true only when the surface  $a$  is flat and when the pressure is the same everywhere.

### 61. Circumferential tension in the walls of a cylindrical pipe.

—The pressure of a fluid in a pipe produces a state of tension in the walls of the pipe. Figure 115 is a side view of a pipe of radius  $r$ . Consider a narrow band of the pipe  $abcd$  of width  $w$ . An end view of this band is shown in Fig. 116. Imagine the fluid in the bottom half of the pipe to be solid as indicated by the shading in Fig. 116, and let us consider the shaded half of the band of pipe (together with its contents) as a definite body  $B$ . This body  $B$  is stationary. Therefore the forces which act upon  $B$  are balanced. But the fluid in the upper half of the pipe pushes

\* At the point where  $\Delta a$  is located.

downwards on  $B$  with a force equal to  $p \times w \times 2r$  as indicated by the short arrows,  $w \times 2r$  being the area of the upper face of  $B$ , and  $p$  being the pressure of the fluid. The band of pipe must,

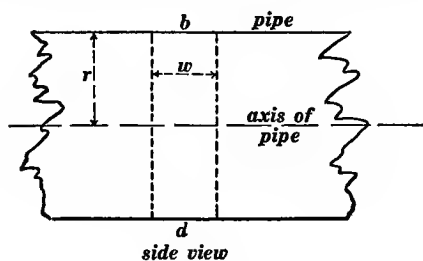


Fig. 115.

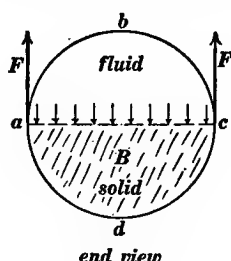


Fig. 116.

therefore, pull upwards on  $B$  to balance this downward push of the fluid. Indeed the two arrows  $FF$  represent the two equal forces with which the band pulls up on  $B$ . Therefore:

$$2F = 2rwp$$

What is called the circumferential tension in the pipe is the force per unit width of the band, namely,  $F/w$ . Therefore, we have:

$$\text{circumferential tension} = \frac{F}{w} = rp$$

**Example.**—A steam pipe is 8 inches in diameter (4 inches in radius) and the pressure of the steam in the pipe is 100 “pounds” per square inch. Therefore the circumferential tension of the material of the pipe is 4 inches multiplied by 100 “pounds” per square inch which gives 400 “pounds” per inch. That is, each circumferential band of the pipe one inch in width is acted upon by a stretching force of 400 “pounds.”

#### PROBLEMS.

110. The piston of a steam engine is 12 inches in diameter, the piston rod is 2 inches in diameter and the steam pressure is 150 “pounds” per square inch. Find the force with which the steam pushes on piston (a) When steam has access to front face

of piston (where piston rod is attached) and (b) when steam has access to back face of piston.

111. Calculate the circumferential tension in the shell of a boiler, the diameter of the boiler being 6 feet and the steam pressure 125 "pounds" per square inch.

112. A given grade of steel can stand safely a tension of 20,000 "pounds" per square inch. What is the greatest diameter of steel tube with walls 0.02 inch thick which can safely withstand a pressure of 150 "pound"s per square inch?

113. Derive an expression for the longitudinal tension (length-wise tension) in the shell of a cylindrical boiler in terms of steam pressure and radius of boiler.

114. Derive an expression for the tension in the shell of a hollow metal sphere in terms of radius of sphere and internal pressure.



62. **Non-uniform pressure in a liquid due to gravity.**—The pressure in any body of fluid at rest would be *uniform* (everywhere the same) if the fluid were *not* acted upon by gravity. This is evident from the following considerations: Imagine a slender cylindrical portion  $PP$  of the fluid as indicated in Fig. 117. The forces exerted on this portion by the surrounding fluid are of course balanced, but the two forces  $F$  and  $F'$  acting on the ends of the cylinder are the only forces parallel to the axis of the cylinder, and since these forces are balanced  $F$  must be equal to  $F'$  or  $ap$  must be equal to  $ap'$  or  $p$  must be equal to  $p'$ ,  $a$  being the sectional area of the cylinder,  $p$  being the pressure of the fluid at one end of the cylinder and  $p'$  being the pressure at the other end of the cylinder.

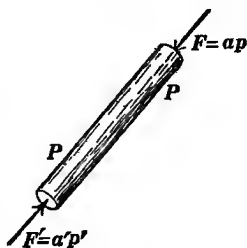


FIG. 117

The pressure in a fluid under the action of gravity increases

with depth. If the density  $d$  of the fluid is everywhere the same (this is very nearly true in a liquid because liquids are very nearly incompressible) then

$$p - p_0 = xdg \quad (16)$$

where  $p_0$  is the pressure at the surface of the liquid,  $p$  is the pressure at a point  $x$  feet (or  $x$  centimeters) beneath the surface,  $d$  is the density of the liquid and  $g$  is the acceleration of gravity.

$p - p_0$  is expressed in "pounds" per square foot when  $x$  is in feet,  $d$  in slugs per cubic foot, and  $g$  in feet per second per second.

$p - p_0$  is expressed in dynes per square centimeter when  $x$  is in centimeters,  $d$  in grams per cubic centimeter, and  $g$  in centimeters per second per second.

See Art. 31 of chapter II. Units of the c.g.s. system or units of the f.s.s. system may be used in all equations in this chapter and in all equations in the following chapter on hydraulics.

**Discussion of equation (16).**—The force with which a liquid pushes on an element of an exposed surface is independent of the direction of the element according to Pascal's principle. Therefore we may derive equation (16) by considering a horizontal surface  $a$  square feet in area exposed to the action of the liquid as shown in Fig. 118. The volume of the liquid directly above

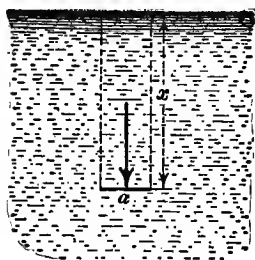


Fig. 118.

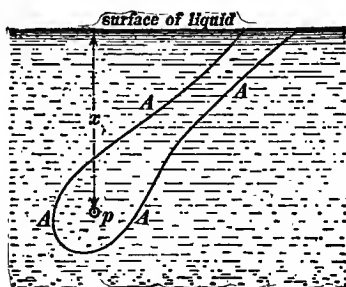


Fig. 119.

$a$  is  $ax$  cubic feet, the mass of this portion of liquid is  $axd$  slugs (where  $d$  is the density of the liquid in slugs per cubic foot), the

force in "pounds" with which gravity pulls on this portion of the liquid is  $axdg$ , and therefore the total force with which this portion of liquid pushes down on the element  $a$  is equal to  $axdg$  "pounds" so that the force per unit area is  $axdg$  divided by  $a$ , or  $x dg$  "pounds" per square foot.

Equation (16) involves no consideration of the shape of the vessel which contains the liquid. As a matter of fact, the pressure at a point in a liquid exceeds the pressure at the surface of the liquid by the amount  $x dg$  whatever the shape and size of the containing vessel may be. This may be made almost self-evident as follows: Given a point  $p$ , Fig. 119, at a distance  $x$  beneath the surface of a large body of liquid. *Imagine a portion of the liquid AAAA, of any shape whatever, extending from  $p$  to the surface.* The liquid surrounding the portion AAAA acts on AAAA exactly as a containing vessel of the same shape would act, and therefore the pressure at  $p$  is exactly what it would be if the portion AAAA were contained in such a vessel.

**63. Total force exerted on a water gate and its point of application.**—Figure 120 represents a body of water of depth  $D$  resting against a vertical gate of width  $w$ . It is desired to find the total force exerted on the gate and the location of the line along which this total or resultant force acts.

Let us consider the pressure of the atmosphere  $p_0$  as zero, then the pressure  $p$  of the water at depth  $x$  will be  $p = x dg$  according to equation (16).

Let  $F$  be the total force acting on the portion of the gate from water surface to depth  $x$  as indicated in Fig. 120. An increase in  $x$  means an increase of  $F$  and the first step in the solution of our problem is to get an expression for  $\frac{dF}{dx}$ .\* Consider the infinitely narrow strip of the gate which is indicated by the black area. The area of this strip is  $w \Delta x$  and the force exerted

\* Do not confuse the  $d$ 's in this symbol with the density  $d$  in the equation  $p = x dg$ .

on this strip is  $p_w \Delta x$  or  $x dgw \Delta x$ . Therefore  $\Delta F = x dgw \Delta x$ , or

$$\frac{dF}{dx} = dgw \cdot x \quad (i)$$

The second step in the solution of our problem is to find a function  $y$  whose derivative  $\frac{dy}{dx}$  is the same as  $\frac{dF}{dx}$ . Let  $y = bx^2$  then  $\frac{dy}{dx} = 2bx$  and if we let  $2b = dgw$  or  $b = \frac{1}{2}dgw$  we will have  $y = \frac{1}{2}dgwx^2$  and  $\frac{dy}{dx} = dgw \cdot x$ . Therefore, according to Art. 27 we have

$$F = \frac{1}{2}dgwx^2 + C$$

where  $C$  is an undetermined constant. Now  $F$  is the total force exerted on the portion of the gate from the surface of the

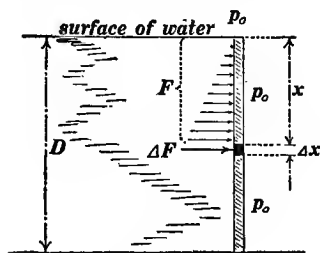


Fig. 120.

The width of the gate (in a direction perpendicular to the plane of the paper) is  $w$ .

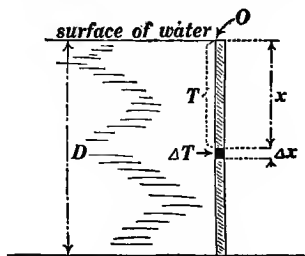


Fig. 121.

water to depth  $x$ , and of course  $F = 0$  if  $x = 0$ , so that the constant  $C$  must be zero, and we have

$$F = \frac{1}{2}dgwx^2$$

where  $F$  is the total force acting on the portion of the gate from the water surface to a depth  $x$ , and if we place  $x = D$  we will get an expression for the total force  $G$  exerted on the entire gate namely

$$G = \frac{1}{2}dgwD^2$$

It is interesting to note that  $G$  is equal to the area of the gate  $wD$  multiplied by the pressure  $\frac{1}{2}dgD$  at the middle point of the gate.

Let  $T$  be the torque action about  $O$ , Fig. 121, of all the forces exerted by the water on the portion  $x$  of the gate. An increase of  $x$  means an increase of  $T$ , and  $\Delta T$  is evidently equal to  $x\Delta F$  where  $\Delta F$  is the force exerted on the portion  $\Delta x$  of the gate. Therefore, using  $\Delta F = xdgw.\Delta x$  we get  $\Delta T = x^2dgw.\Delta x$  or

$$\frac{dT}{dx} = dgw.x^2 \quad (ii)$$

Now it can be easily shown that  $y = \frac{1}{3}dgwx^3$  is a function whose derivative  $\frac{dy}{dx}$  is the same as  $\frac{dT}{dx}$  so that we get

$$T = \frac{1}{3}dgwx^3 + C$$

but  $T = 0$  when  $x = 0$  so that  $C$  must be zero. Therefore

$$T = \frac{1}{3}dgwx^3$$

where  $T$  is the torque action about  $O$  of all the forces which act on the portion  $x$  of the gate, and if we put  $x = D$  we will get the torque action  $T'$  about  $O$  of the forces acting on the entire gate, namely

$$T' = \frac{1}{3}dgwD^3$$

The line of action of the total or resultant force  $G$  must be at such distance  $l$  beneath  $O$  that the torque action  $lG$  may be equal to  $T'$ . Therefore from  $lG = \frac{1}{3}dgwD^3$ , using  $G = \frac{1}{2}dgwD^2$  we get

$$l = \frac{2}{3}D$$

The total or resultant force  $G$  is, of course, a horizontal force, and its line of action is  $\frac{2}{3}D$  below the water surface.

#### PROBLEMS.

**115.** The water pressure at a fire hydrant in a city is 125 "pounds" per square inch at 2 A.M. when the water in the street mains is not moving perceptibly. Calculate the height  $h$  of the water level in the city reservoir above the fire hydrant.

*Note.* Weight (true weight) in pull-pounds per cubic foot is  $w = dg$ , where  $d$  is density in slugs per cubic foot and  $g$  is the local acceleration of gravity. Therefore equation (16) may be written

$$p - p_0 = xw$$

and for most practical purposes  $w$  may be considered as numerically equal to density in sugar-pounds per cubic foot. This is equivalent to taking the acceleration of gravity everywhere to be the same as at London.

116. What air pressure is required to hold the water level in a caisson at a point 62 feet below the water level in a river?

117a. What is the total force exerted by the liquid on the end of a rectangular tank which is 10 feet wide  $\times$  12 feet deep, the tank being full of syrup of which the density is 80 pounds per cubic foot?

*Note.*—Take atmospheric pressure as zero.

117b. One end of a large rectangular water tank has a circular hole in it and the hole is 5 feet in diameter with its center 7 feet beneath the water surface. What is the total force exerted on a board covering the hole?

*Note.*—The *average pressure* over any submerged flat surface is the pressure at the center of figure ("center of gravity") of the flat surface.

118. A rectangular tank 10 feet wide  $\times$  16 feet long  $\times$  12 feet deep is filled  $\frac{1}{3}$  full of water (density  $62\frac{1}{2}$  pounds per cubic foot) and the remainder of the tank contains oil of which the density is 56 pounds per cubic foot. What total force is exerted on one end of the tank by the water?

*Note.*—Take atmospheric pressure as zero.



64. The **barometer** is an instrument for measuring the pressure of the atmosphere. It consists of a tube  $T$ , Fig. 122, filled with mercury and inverted in an open vessel of mercury as indicated. An empty space  $V$  is left in which the pressure is zero.\*

The pressure of the mercury *in the tube at the level of the outside mercury surface* is equal to the pressure of the air, and it exceeds the pressure in  $V$  by the amount  $xdg$ , according to equation (16) where  $x$  is the height of the mercury column in the tube as shown in the figure,  $d$  is the density of the mercury,

\* Even if the tube is filled with great care (mercury boiled in the tube as it is filled), mercury vapor will form in the region  $V$  and the pressure in this region will not be quite zero.

and  $g$  is the acceleration of gravity. This expression  $xdg$  gives the value of the atmospheric pressure in dynes per square centimeter (or in "pounds" per square foot) as explained in Art. 62.

If the mercury is at some standard temperature,  $d$  is invariable; and if the barometer is used in a given locality,  $g$  is invariable. *Under these conditions the distance  $x$  may be used as a measure of the pressure.* In fact, atmospheric pressure is usually expressed in terms of the height the barometric column would have in millimeters or in inches if the mercury were at  $0^\circ$  C. and if the acceleration of gravity were 981.61 centimeters per second per second (its value at  $45^\circ$  north latitude at sea level).

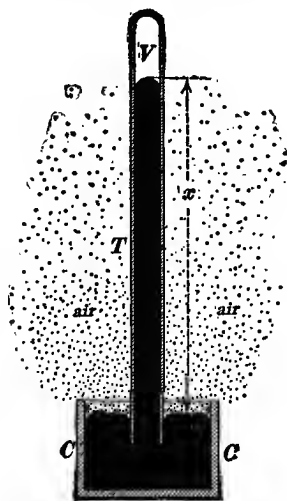


Fig. 122.

**65. Buoyant force of fluids.**—The pressure in a fluid increases with depth as explained in Art. 62. Therefore the upward forces exerted on the lower surface of a submerged body are greater than the downward forces exerted on the upper surface of the body as shown in Fig. 110. Therefore, on the whole, a fluid exerts an upward force on a submerged body. This upward force is called the *buoyant force* of the fluid on the body.

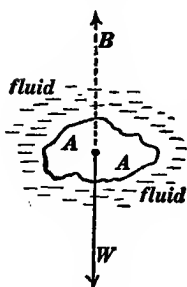


Fig. 123.

**Archimedes' principle.**—*The buoyant force exerted by a fluid on a submerged body of volume  $V$  is equal to the weight of the same volume of the fluid.* This fact is called *Archimedes' principle* from its discoverer, and it may be made almost self-evident from the following considerations. Given a fluid at rest, and let us think of a definite portion  $AA$  of this fluid of any size and shape,

as shown in Fig. 123. This portion  $AA$  is stationary, and therefore the downward pull of gravity on  $AA$  (the weight  $W$  of  $AA$ ) must be balanced by the total\* force  $B$  with which the surrounding fluid pushes upwards on  $AA$ . That is  $B$  must be equal and opposite to  $W$ , and  $B$  and  $W$  must lie in the same straight line. But the surrounding fluid pushes on  $AA$  exactly as it would on any submerged body of the size and shape of  $AA$ . Therefore  $B$  is the buoyant force of the fluid on any body of the size and shape of  $AA$ , and as stated above,  $B$  is equal in value to the weight  $W$  of the portion  $AA$  of fluid.

**Archimedes' principle as applied to a body floating on a liquid.**—The buoyant force of a liquid on a floating body is equal (and opposite) to the downward pull of gravity on the body (the weight of the body) because the floating body remains stationary. But the buoyant force of a liquid on a floating body is equal to the weight of the liquid displaced by the body. Therefore *a floating body displaces its weight of the liquid in which it floats.*

**Examples.**—A balloon has a total volume of 400 cubic meters, and the balloon and contained hydrogen have a mass of 286 kilograms. The balloon, of course, displaces 400 cubic meters of air, and the density of air is 1.2 kilograms per cubic meter, so that the balloon displaces 480 kilograms of air. Therefore the total buoyant force of the air on the balloon is equal to the weight of 480 kilograms of material, and the downward pull of the earth on the balloon is of course equal to its weight (the weight of 286 kilograms of material). Consequently the upward push of the air on the balloon exceeds the weight of the balloon by an amount equal to the weight of 480 kilograms *minus* 286 kilograms or 194 kilograms of material.

A boat displaces 2,000 cubic feet of water so that the buoyant force of the water on the boat (which is equal to the weight of the boat) is equal to the weight of 2,000 cubic feet of water, or to the weight of 125,000 pounds of water.

\* The fluid exerts a push on each element of the surface of  $AA$ , and these pushes are together equivalent to the single force  $B$  which is called their vector sum or resultant.

**66. Density and specific gravity.**—The *density* of a substance is its mass per unit volume, that is

$$D = \frac{M}{V}$$

where  $M$  is the mass of a body in grams,  $V$  is its volume in cubic centimeters and  $D$  is its density in grams per cubic centimeter.

The *specific gravity* of a substance at a given temperature is the ratio of the density of the substance to the density of water at the same temperature. Thus, to say that the specific gravity of iron is 7.78 means that the density of iron is 7.78 times the density of water, or it is equivalent to saying that the mass of a given volume of iron is 7.78 times the mass of the same volume of water.

In some cases a great deal of tedious circumlocution is required to distinguish sharply between *weight* and *mass*. The pull of the earth on a one-gram body (or on a one-pound body) *at any given place* is a perfectly definite force, and if we express the *weight* of a body in terms of this unit of force, then the weight of a body will be the same thing numerically as its mass. This simple scheme is adopted in the following discussion.

(a) The volume of a vessel can be very accurately determined by weighing the empty vessel and then weighing the vessel full of water or mercury. The difference is the net weight of the water or mercury, and by dividing this by the density of the water or mercury the volume of the vessel is found. *Example.* A vessel weighs 286.52 grams empty and 948.23 grams when filled with water at 20° C. Therefore the net weight of the water is 661.71 grams, which divided by 0.9983 grams per cubic centimeter (the density of water at 20° C.) gives 663 cubic centimeters as the volume of the vessel.

(b) A substance having a volume of  $V$  cubic centimeters weighs 546.2 grams in air\* and 432.6 grams when suspended by a thread and submerged in water at 20° C. The difference is

\*Buoyant force of air is neglected in these examples.

the buoyant force of the water, which is the weight of  $V$  cubic centimeters of water. Therefore the weights of equal volumes of the substance and of water are 546.2 grams and 113.6 grams respectively, so that the *specific gravity* of the substance at  $20^{\circ}$  is  $546.2 \text{ grams} \div 113.6 \text{ grams}$  or 4.808. That is, the substance is 4.808 times as heavy as water. But the density of water at  $20^{\circ}$  C. is 0.9983 gram per cubic centimeter, and therefore the *density* of the substance is  $4.808 \times 0.9983$  grams per cubic centimeter or 4.6998 grams per cubic centimeter.

(c) A glass ball weighs 72.44 grams in air, 45.22 grams in water, and 47.94 grams in oil. The loss of weight of the ball in water (27.22 grams) is the weight of its volume of water, and the loss of weight of the ball in the oil (24.50 grams) is the weight of its volume of oil. Therefore the specific gravity of the oil is  $24.50 \text{ grams} \div 27.22 \text{ grams}$  or 0.900. That is, the oil is 0.900 as heavy as water.

**67. Cohesion; adhesion.**—When a body is under stress, as for example a stretched wire, the tendency of the stress is to tear the contiguous parts of the body asunder. The forces which oppose this tendency and hold the contiguous parts of a body together are called the forces of *cohesion*. The forces which cause dissimilar substances to cling together are called the forces of *adhesion*. The cohesion of water and the adhesion between water and glass are the forces which determine the curious behavior of water in a fine hair-like tube of glass, and the phenomena exhibited by liquids because of cohesion and adhesion are called *capillary phenomena* from the Latin word *capillaris* meaning hair.

**68. Surface tension.**—On account of their cohesion, all liquids behave as if their free surfaces were stretched skins, that is, as if their free surfaces were under tension. Thus a drop of a liquid tends to assume a spherical shape on account of its surface tension. A mixture of water and alcohol may be made of the same density as olive oil, and a drop of olive oil suspended in such a mixture becomes perfectly spherical.

Many curious phenomena\* are produced by the variation of the surface tension of a liquid with admixture of other liquids or with temperature. Thus a drop of kerosene spreads out in an ever widening layer on a clean water surface on account of the fact that the tension of the clean water surface beyond the layer of oil is greater than the tension of the oily surface. A small shaving of camphor gum darts about in a very striking manner upon a clean water surface, on account of the fact that the camphor dissolves in the water more rapidly where the shaving happens to have a sharp projecting point, the water surface has a lessened tension where the camphor dissolves, and the greater tension on the opposite side pulls the shaving along. A minute cork boat with a small bit of camphor gum fixed to its stern is pulled along for the same reason. A thin layer of water on a horizontal glass plate draws itself away and leaves a dry spot where a drop of alcohol is let fall on the plate. A thin layer of lard on the bottom of a frying pan pulls itself away from the hotter parts of the pan and heaps itself up on the cooler parts, because of the greater surface tension of the cooler lard.

## PROBLEMS.

119. The density of mercury at  $0^{\circ}$  C. is 13.5956 grams per cubic centimeter. Calculate the value in dynes per square centimeter of standard atmospheric pressure, namely 76 cm. of mercury at  $0^{\circ}$  C., the value of gravity being 980.61 cm. per second per second. Ans. 1,012,900 dynes per square centimeter.

120. Calculate the height of the *homogeneous atmosphere*; that is, assuming that the atmosphere has a uniform density of 0.00129 gram per cubic centimeter throughout, calculate the depth which would produce standard atmospheric pressure. Ans. 8,012 meters or 4.98 miles.

*Note.*—Take 980 centimeters per second per second for the acceleration of gravity.

\* See the very interesting article on *capillary action* in the 9th edition of the Encyclopedia Britannica. This article gives a comprehensive discussion of the theory of capillary action.

**121.** What is the net lifting capacity of a balloon containing 550 cubic meters of hydrogen, mass of balloon material being 275 kilograms. Density of air = 1200 grams per cubic meter, density of hydrogen = 90 grams per cubic meter.

**122.** The density of sea water is about 1.02 times the density of pure water. Find total volume of sea water displaced by a ship having a total mass of 25,000 tons (1 ton equal to 2,000 pounds). The area of the water level section of the ship is 12,000 square feet. How much will the ship sink on entering a basin of fresh water (density 62.4 pounds per cubic foot)?

**123.** A crystal of alum weighs 125 grams in air and 60.0 grams when submerged in an oil whose density is 0.910 gram per cubic centimeter. What is the density of the alum crystal?

**124.** How many grams of aluminum (density 2.7 grams per cubic centimeter) must be fastened to 25 grams of cork (density 0.28 gram per cubic centimeter) so that the whole will barely sink in an oil whose density is 0.91 gram per cubic centimeter?

**125a.** A tub of water standing on a platform scale weighs 150 pounds. A 250-pound block of iron (specific gravity 7.78) is hung by a rope and submerged in the water in the tub. Find apparent increase of weight of tub and water.

**125b.** A metal can 20 centimeters in diameter floats with 25 centimeters of its length out of water when a 6000-gram iron sinker is hung from the bottom of the can. How much of the can will be above the water if the iron sinker is placed inside of the can? Specific gravity of iron 7.78.

## CHAPTER V.

### HYDRAULICS.

**69. Subject and limitations of this chapter.**—Hydraulics, in the general sense in which the term is here used, is the study of liquids and gases in motion; and the phenomena which are presented in this branch of physics are excessively complicated. Even the apparently steady flow of a river through a smooth sandy channel is an endlessly intricate combination of boiling and whirling motion; and the jet of spray from a hydrant, or the burst of steam from the safety-valve of a locomotive, what is to be said of things such as these? Or let one consider the fitful motion of the wind as indicated by the swaying of trees and as actually visible in driven clouds of dust and smoke, or the sweep of the flames in a conflagration! These are *actual* examples of fluids in motion, and they are indescribably, infinitely\* complicated

*The science of hydraulics is based on ideas which refer to average aspects of fluid motion.*—Thus the engineer is concerned chiefly with such things as the time required to draw a pail of water from a hydrant, the loss of pressure in a line of pipe between a pump and a fire nozzle, or the force exerted by a water jet on the buckets of a water wheel. These are called average effects because they are never perfectly steady but always subject to perceptible fluctuations of an erratic character, and to think of any

\* Everyone concedes the idea of infinity which is based upon abstract numerals (one, two, three, four and so on *ad infinitum*!), and the idea of infinity which is based on the notion of a straight line; but most men are wholly concerned with the humanly significant and persistent phases of the material world, their perception does not penetrate into the substratum of utterly confused and erratic action which underlies every physical phenomenon, and they balk at the suggestion that the phenomena of fluid motion, for example, are infinitely complicated. Surely the abstract idea of infinity is as nothing compared with the intimation of infinity that comes from things that are seen and felt.

of these effects as having a *definite value* is, of course, to think of its *average value* under the given conditions. The extent to which the practical science of hydraulics is limited by the consideration of average effects is evident from the following outline of the ideal types of flow upon which nearly the whole of the science is based.

**Permanent and varying states of flow.**—When a hydrant is suddenly opened, it takes an appreciable time for the flow of water to become steady. During this time (*a*) *the velocity at each point of the stream is increasing and perhaps changing in direction also*. After a short time, however, the flow becomes fully established and then (*b*) *the velocity at each point in the stream remains unchanged in magnitude and direction*.\* The motion (*a*) is called a *varying state of flow*, and the motion (*b*) is called a *permanent state of flow*. Most of the following discussion applies to permanent states of flow, indeed there are but few cases in which it is important to consider varying states of flow.

**The idea of simple flow. Stream lines.**—The idea of simple flow applies both to permanent and to varying states of flow but it is sufficient to explain the idea in its application to permanent flow only. When water flows steadily through a pipe, the motion is always more or less complicated by continually changing

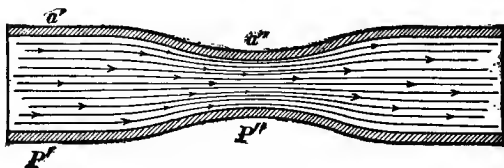


Fig. 124

eddies, the water at a given point *does not* continue to move in a fixed direction at a constant velocity; nevertheless, it is convenient to treat the motion as if the velocity of the water were in a fixed direction and of constant magnitude at each point. Such

\* Assuming the stream to be free from turbulence. See the following definition of simple flow.

a motion is called a *simple flow*. In the case of a simple flow, a line can be imagined to be drawn through the fluid so as to be at each point in the direction of the flow at that point. Such a line is called a *stream line*. Thus the fine lines in Fig. 124 are stream lines representing a simple flow of water through a contracted part of a pipe. To apply the idea of simple flow to an actual case of fluid motion is the same thing as to consider the average character of the motion during a fairly long interval of time.

**Lamellar flow.**—Even though the motion of water in a pipe may be approximately a simple flow, the velocity may not be the same at every point in a given cross-section of the pipe, that is, the velocity may not be the same at every part of the layer *ab*, Fig. 125; in fact the water near the walls always moves slower

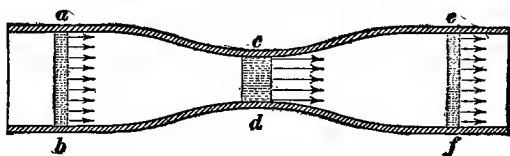


Fig. 125.

than the water near the center of the pipe; nevertheless, it is convenient in many cases to treat the motion as if the velocity were the same at every point in any layer like *ab*, Fig. 125. Such an ideal flow is called a *lamellar flow*, because in such a flow the fluid in any layer or lamella *ab* would later be found in the layer *cd*, and still later in the layer *ef*. To apply the idea of lamellar flow to an actual case of fluid motion is the same thing as to consider the average velocity over the entire cross-section of a stream.

70. Some phenomena of fluid motion not associated with permanent, simple, lamellar flow.—The theoretical treatment of fluid motion in this chapter is so largely based on the ideas of permanent, simple, lamellar flow that we shall be carried far away from many interesting phenomena of fluid motion.

The action of the hydraulic ram is a good example of varying

flow. Water flows from a low dam through a long pipe  $PP$  and escapes through an open valve  $A$  as indicated in Fig. 126. The

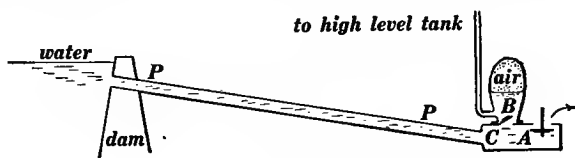


Fig. 126.

flowing water lifts the valve  $A$ , thus suddenly closing it, and the hammer or ram action of the moving water in  $PP$  drives a small quantity of water through the check valve  $B$  and into a high level tank. The water in  $PP$  is thus brought to rest with excessive pressure in  $CA$  which causes the column of water  $PPCA$  to rebound, thus momentarily reducing the pressure in  $CA$  and opening the valve  $A$  as at first, and the above described action is repeated.

The tendency of a stream of fluid to become turbulent (to depart from ideal simple flow) is exemplified by the sensitive flame.—When a fluid flows *slowly* through a channel or pipe, or as a jet through a body of surrounding stationary fluid, the motion approximates very closely to a *simple flow*. When the velocity of the fluid is increased, however, a critical velocity is soon reached at which the flow suddenly becomes very *turbulent* (full of eddies). This sudden increase of turbulence is illustrated by the behavior of an ordinary gas flame (the flame serves only to make the jet of gas visible). When the gas is turned on slowly the flame is at first smooth and steady, but a certain point is reached (a certain velocity of the gas in the jet) at which the flame suddenly becomes rough and unsteady, innumerable eddies form at the boundary between the moving gas and the still air. When the flame is on the verge of becoming unsteady it is sometimes extremely sensitive, the least hissing sound causes it to become turbulent.

The boundary between the moving gas and the still air con-

stitutes what is called a *vortex sheet*, and the behavior of the sensitive flame is due to the instability of a vortex sheet; any disturbance, however small, starts a minute eddy which develops more and more.

The behavior of the so-called spit ball is due to the instability of the vortex sheet.—A perfectly smooth spherical ball (not spinning) moves forwards through still air, and, since everything is symmetrical with respect to the line of motion, there can be no reason why the ball should jump to the right rather than to the left, therefore we may conclude that the ball will not jump either way! But the ball *does* jump sidewise as in case of the so-called *spit ball*, and a marble dropped in a jar of water follows an irregular zigzag path as indicated by the dotted line in Fig. 127.

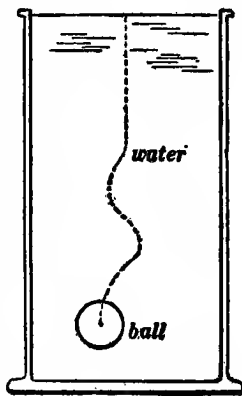


Fig. 127.

A sharp pointed stick stands vertically in a still room, and, since everything is symmetrical with respect to the axis of the stick, there can be no reason why the stick should fall one way rather than another, therefore we may conclude that the stick will not fall either way! But the stick *does* fall. The vertical stick is in a condition of instability, and any disturbance, however small, starts the stick falling, and a fall once started develops more and more, as every one knows.

Figure 128 shows the air blowing past a ball (as if the ball were

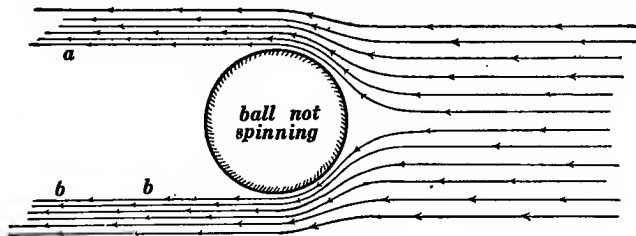


Fig. 128.

stationary and the air moving) and breaking away from the ball so as to leave a body of still air back of the ball. We thus have a vortex sheet  $aa$  and  $bb$ , and if the velocity of the air (velocity of the ball if the air is still) is just right this vortex sheet is unstable. Any disturbance, however small, starts an eddy which develops more and more. As a result there is a sidewise fluttering of the air stream back of the ball, and the reaction of this fluttering stream pushes the ball to one side and to the other irregularly. This effect is familiar to anyone who has held a thin stick in the moving water at one side of a row boat or launch, the fluttering vortex sheets back of the stick react on the stick and give to it a quivering motion.

**The distinction between lamellar flow and non-lamellar flow** in a pipe or channel is a simple example of a more fundamental distinction between *irrotational flow* and *rotational flow*, between a kind of flow in which the individual particles of the fluid do not rotate and a kind of flow in which the individual particles of the fluid do rotate. This distinction is one of great importance in the mathematical theory of fluid motion.\* **The familiar smoke ring and the whirlpool in an emptying wash bowl are examples of rotational fluid motion.**

When water flows out of a hole in the bottom of a bowl a whirlpool generally forms above the hole. *The formation of this whirlpool depends upon the previous existence of a slow rotatory motion of the water in the bowl*, which rotatory motion is greatly increased when the water flows towards the hole. This increase of spin velocity of a body as the parts of the body move towards the axis of spin is strikingly illustrated by the following experiment. A person standing on a pivoted stool is set spinning about a vertical axis with his arms outstretched and with weights in his hands; and when he draws the weights in towards his body (towards the axis of spin) his spin velocity is greatly increased.†

\* See Franklin, MacNutt and Charles' *Calculus*, pages 242-250.

† The spin-momentum  $Ks$  of a rotating body never changes unless an outside torque acts on the body. Therefore, if the spin-inertia  $K$  of the rotating body is decreased, its spin velocity  $s$  must increase.

The rotation of the earth on its axis involves a slow turning of one's horizon about a vertical axis (except at the equator). When the air near the ground is warmed by the sun's rays it starts to flow upwards at some place, a chimney-like effect is produced by the rising column of warm air, the lower layer of warm air is drawn towards this central chimney from all sides and the slow turning motion of one's horizon becomes greatly increased as a more or less violent whirl at the central chimney. The *cyclone*\* is a movement of this kind covering thousands of square miles of country with a central chimney hundreds of miles in diameter. The *tornado* is a movement of this kind covering only a few square miles of country with a central chimney only a few hundred yards in diameter. The tornado is often very violent and destructive.

**71. Rate of discharge of a stream.**—The volume of water which is delivered per second by a stream is called the *discharge rate* of the stream. Thus the mean discharge rate of the Niagara River is 300,000 cubic feet per second. *The rate of discharge of a stream is equal to the product of the average velocity,  $v$ , of the stream and the sectional area,  $a$ , of the stream.* For example, let  $PP$ , Fig. 129, be the end of a pipe out of which water is flowing and let us assume that the velocity of flow has the same value  $v$  over the entire section of the stream (lamellar flow), then the water which flows out of the end of the pipe in  $t$  seconds would make a cylinder or prism of length  $vt$ , and of sectional area  $a$ , as indicated in the figure, and the volume of this water is therefore  $avt$ . Dividing this volume by the time  $t$  gives the discharge rate  $av$ .

*Variation of velocity with sectional area of a steady lamellar stream.* Consider a simple flow of water through a pipe as indicated by the stream lines in Fig. 124. Let  $a'$  and  $a''$  be the cross-sectional areas of the stream at any two points  $P'$  and  $P''$ , and let  $v'$  and  $v''$  be the average or lamellar velocities of the stream at  $P'$  and  $P''$  respectively. Then  $a'v'$  is the

\* What is popularly called a *cyclone* is properly called a *tornado*.

volume of water which passes the point  $P'$  per second, and  $a''v''$  is the volume of water which passes the point  $P''$  per

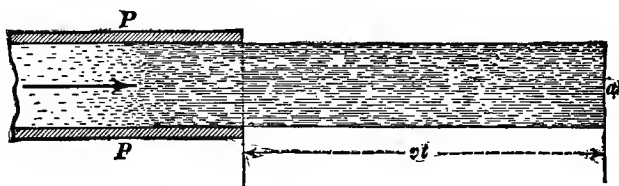


Fig. 129.

second; and, therefore, since the same amount of water must pass each point per second, we have

$$a'v' = a''v'' \quad (17)$$

that is, the product  $av$  has the same value all along the pipe, so that  $v$  is large where  $a$  is small, and  $v$  is small where  $a$  is large.

Equation (17) applies only to a fluid which is approximately incompressible like water or any other liquid. In such a case  $a'v'$  is the amount of water per second entering one end of a pipe and  $a''v''$  is the amount of water per second flowing out of the other end of the pipe, and these two expressions must be equal to each other. If, however, the fluid is compressible like a gas, then equation (17) becomes

$$a'v'd' = a''v''d''$$

where  $a'$  is the sectional area of the steady stream of gas at one place,  $v'$  is the average velocity of the stream at that place,  $d'$  is the density of the gas at that place, and  $a''$ ,  $v''$  and  $d''$  are the cross-sectional area, the velocity of the stream and the density of the gas at another part of the stream.

**72. The ideal frictionless, incompressible fluid.**—When a jet of water issues from a tank, there is a certain relation between the velocity of the jet and the difference in pressure inside and outside of the tank. When there are variations of the velocity of flow of water along a pipe due to enlargements or contractions of the pipe [see equation (17)], the pressure decreases wherever the velocity increases and *vice versa*. These mutually dependent changes of velocity and pressure are always complicated by friction, and by the variations of the density of the fluid due to the variations of pressure; and in order to gain the simplest

possible idea of these mutually dependent changes of velocity and pressure the conception of the *frictionless incompressible fluid* is very useful.

When the water in a pail is set in motion by stirring, it soon comes to rest when left to itself. *A fluid which would continue to move indefinitely after stirring would be called a frictionless fluid.*

When a moving fluid is brought to rest by friction, the kinetic energy of the moving fluid is converted into heat and lost. Such a loss of energy would not take place in a frictionless fluid, and therefore the total energy (kinetic energy plus potential energy) of a frictionless fluid would be constant. *This principle of the constancy of total energy is the basis of the following discussion (Arts. 73 and 74). This discussion applies in all strictness to the ideal frictionless, incompressible fluid, only; but the formulas are extensively used by engineers, nevertheless.*

### 73. Energy of a liquid.—(a) Potential energy per unit of volume.

An incompressible liquid under pressure in a tank represents a store of potential energy only when the pressure in the tank is maintained while the liquid flows out of the tank.\* This condition is realized, for example, in a tank which is connected to a high-level reservoir, as in a city water-system. The following discussion of the potential energy of an incompressible liquid may therefore be thought of as applying to such a tank.

Figure 130 represents the piston  $CC$  of a pump which is forcing water into a tank at pressure  $p$ . The force which must push on the piston (ignoring friction) is  $ap$ , where  $a$  is the area of the piston. An amount of work  $apD$  is done in pushing the piston over a distance  $D$ . Therefore to pump volume  $aD$  of liquid into the tank involves the doing of  $apD$  units

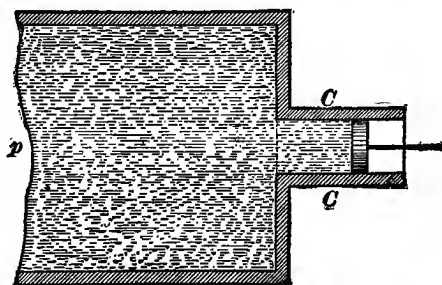


Fig. 130.

\* The pressure of an ideally incompressible liquid in an ideally rigid tank would drop to zero with the outflow of an infinitesimal amount of the liquid.

of work, which is  $p$  units of work for each unit of volume. Therefore the potential energy of the liquid in the tank per unit of volume is

$$W' = p \quad (i)$$

(b) *Kinetic energy per unit volume.* Let  $v$  be the velocity of a moving liquid and let  $d$  be the mass of unit volume (the density) of the liquid. Then the kinetic energy of unit volume of the liquid, according to equation (3) of Art. 37, is

$$W'' = \frac{1}{2}dv^2 \quad (ii)$$

#### PROBLEMS.

**126.** Find the lamellar velocity at which the water must flow in a canal 20 feet wide and 6 feet deep in order that the discharge rate may be 500 cubic feet per second.

**127.** A water main 8 inches inside diameter discharges 600 gallons of water per minute (1 gallon = 231 cubic inches). Find the lamellar velocity of the water in the pipe.

**128.** The lamellar velocity of the water in a pipe is 6 feet per second where the pipe is 1 foot inside diameter. What is the lamellar velocity of the water at a place where the inside diameter of the pipe is reduced to 7 inches?

**129.** How much work is required to pump 60 cubic feet of water into a tank at gage-pressure of 100 "pounds" per square inch, the water to be lifted 20 feet at the same time? Neglect the friction in the pump.

*Note.* Gage-pressure means pressure reckoned above the pressure of the atmosphere.

**130.** A stream of water has a velocity of 12 feet per second. What is its kinetic energy per cubic foot? How high would the water have to be lifted in order that its potential energy per cubic foot may be the same as its potential energy per cubic foot in the stream? How much gage-pressure would there have to be in a tank in order that the potential energy of the water per

cubic foot may be the same as its kinetic energy per cubic foot in the stream?

**74. Bernoulli's Principle.** Equation expressing constancy of total energy of a frictionless, incompressible fluid.—Figure 131 represents a pipe or channel (like Fig. 124) through which a frictionless, incompressible fluid is flowing. At *A* the pressure

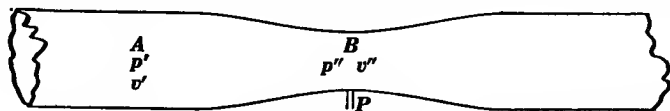


Fig. 131.

is  $p'$  and the velocity is  $v'$ , and at *B* the pressure is  $p''$  and the velocity is  $v''$ . Consider a portion or "chunk" of the fluid as it flows along the pipe or channel. When the chunk is at *A* its total energy per unit volume is  $p' + \frac{1}{2}dv'^2$  according to equations (i) and (ii) of Art. 73, and when the chunk is at *B* its total energy per unit volume is  $p'' + \frac{1}{2}dv''^2$ . If the liquid were frictionless the total energy would not change so that we would have:

$$p' + \frac{1}{2}dv'^2 = p'' + \frac{1}{2}dv''^2 \quad (18)^*$$

In this equation c.g.s. units must be used throughout, or f.s.s. units must be used throughout. See Art. 31 of Chapter II.

Equation (18) shows that if  $v'$  is larger than  $v''$  then  $p'$  must be smaller than  $p''$  and *vice versa*. This relation was discovered by John Bernoulli and it is known as *Bernoulli's principle*.

**Limitations of Bernoulli's principle.**—Bernoulli's principle as expressed by equation (18) applies only to frictionless, incompressible fluids, but it is approximately true for ordinary liquids

\* This equation applies only to a horizontal stream. Let  $x'$  be the altitude of point *A* above a chosen base and let  $x''$  be the altitude of point *B* above the chosen base. Then equation (18) becomes:

$$p' + x'dg + \frac{1}{2}dv'^2 = p'' + x''dg + \frac{1}{2}dv''^2$$

and gases when friction is not excessive and where changes of pressure (and consequent changes of density) are not great.

Bernoulli's principle applies only to permanent states of flow.

Bernoulli's principle applies only to what is called irrotational flow. Thus in a rapidly rotating bowl the pressure is greatest near the periphery of the bowl where the velocity is greatest. As another example of rotational flow consider Fig. 128. The existence of the vortex sheet  $aa\ bb$  means that the motion is rotational and the pressure in the still fluid on one side of the vortex sheet is approximately the same as the pressure in the moving fluid on the other side of the vortex sheet.

QUALITATIVE EXAMPLES OF BERNOULLI'S PRINCIPLE. (a) *The disk paradox.*—

Figures 132a and 132b represent a short piece of brass tube  $TT$  with a flat brass disk  $DD$  fixed to its end, and  $dd$  is a light metal disk. When one blows through the tube  $TT$ , the disk  $dd$  is not blown away from  $DD$ , but the outside air pushes  $dd$  very strongly against  $DD$  because of the low pressure of the rapidly moving air between the two disks.

Figure 132b shows a top view of the arrangement, and the small arrows in Fig. 132b represent the air blowing out from between the edges of the disks. Consider the air stream between the disks, its sectional area increases towards the edge of the disks. In fact  $c'h$  is the sectional area of the stream at the dotted circle  $c'$  and

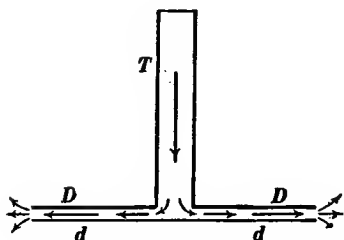


Fig. 132a.

Side view.

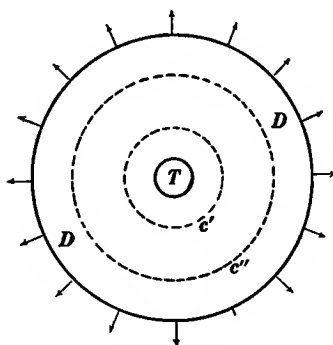


Fig. 132b.

Top view.

$c''h$  is the sectional area of the stream at the dotted circle  $c''$ ,  $h$  being the distance between the disks (a constant) and  $c'$  and  $c''$  being the circumferences of the respective dotted circles. Evidently  $c''h$  is larger than  $c'h$ . Therefore the velocity of the air stream decreases towards the edges of the disks, according to

equation (17) of Art. 71, and consequently the pressure of the air between the disks increases towards the edges of the disks.

But the pressure of the air at the edges of the disks is atmospheric pressure. Therefore the pressure of the air between the disks is everywhere less than atmospheric pressure, so that the outside air pushes the disks together.

The pressure of the air in the tube *TT* is not considered.

(b) *The jet pump.*—The pressure in the throat *B* in Fig. 131 is less than the pressure at *A*; indeed the pressure at *B* may be small enough to suck liquid into the throat through the side tube *P*. Liquid thus sucked into the throat is carried along with the main stream in the pipe. Such an arrangement is called a *jet pump*.

(c) *Ship suction.*—Two ships steaming along side by side are drawn together by the action of the water. In this case a given particle of the water (as indicated by a small float) is stationary when the boats are far distant, it moves slightly as the boats pass by, and then comes to rest again. The water in the neighborhood of the two moving boats is not in a permanent state of flow, so that, as it would seem, Bernoulli's principle cannot be applied. But the force action between the water and the boats depends only on the velocity of the boats relative to the water, or on the velocity of the water relative to the boats. Therefore we may think of the boats as standing still with the water flowing steadily past them as indicated by the stream lines in Fig. 133. From this point of view the water is in a permanent

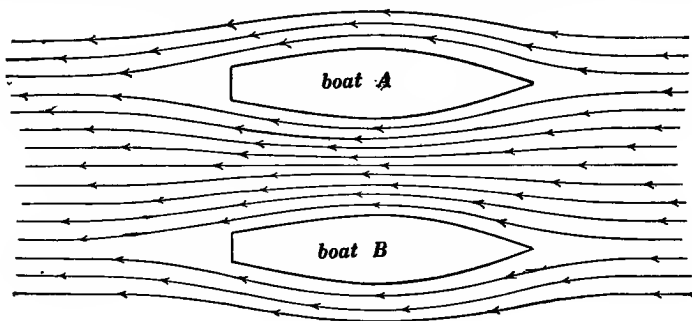


Fig. 133.

state of flow and Bernoulli's principle may be applied. The stream lines are greatly crowded together in the region between the boats, and only slightly crowded together in the regions on the outer sides of the two boats. Therefore the velocity of the water is greater\* between the boats than on the outer sides of the boats. Therefore the water level† is lower between the boats than on the outer sides of the two boats, and the higher level water on the outer sides pushes the boats together.

(d) *The curved flight of a spinning baseball.*—In order to be able to apply Bernoulli's principle to the air in the neighborhood of a moving ball we must think of

\* According to equation (17) of Art. 71.

† Change of level corresponds to change of pressure.

the ball as standing still (spinning or not as the case may be) with the air blowing past it. The curved lines in Fig. 134 represent a stream of air flowing past a ball which is not spinning, and the fine circles in Fig. 135 represent the motion of the

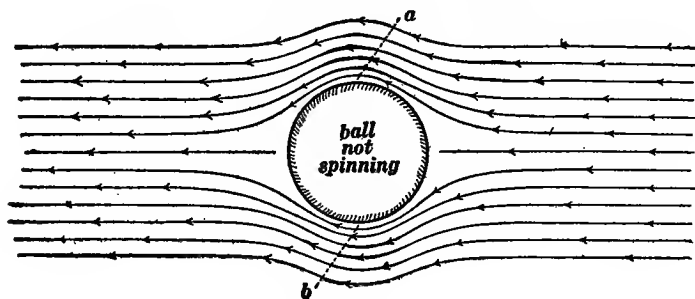


Fig. 134.

Blast of air blowing past a ball which is not spinning.

air in the neighborhood of a spinning ball, the air being still except for the effect of the spinning ball. When a blast of air blows past a spinning ball as shown in

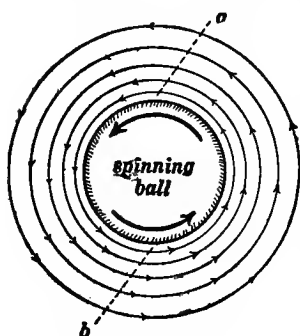


Fig. 135.

Whirl of air in the neighborhood of a spinning ball.

Fig. 136 the actual velocity of the air at *a* is the *sum* of the velocities at *a* in Figs. 134 and 135; and the actual velocity of the air at *b* in Fig. 136 is the *difference* of the velocities at *b* in Figs. 134 and 135. This is evident when we consider that the motion of the air in Fig. 136 is due to the cause of air motion in Fig. 135 and the cause of air motion in Fig. 135 *both acting together*. Therefore the velocity of the air is great at *a* and small at *b* in Fig. 136, and therefore, according to Bernoulli's principle, the pressure of the air is great at *b* and small at *a*, and consequently the air underneath the ball in Fig. 136 pushes upwards on the ball with a greater force than the air above the ball pushes downwards on it. These two forces are represented by *F* and *f* in Fig. 137.

The force action of the air on the ball in Fig. 136 is the same as that which would exist if the spinning ball were moving forwards through still air as indicated in Fig. 137; and the greater force *F* causes the ball to be deflected, as indicated by the curved arrow, from the direction in which it would continue to travel if it were not spinning.

Let us call the foremost point, *N*, of the ball the *nose* of the ball. It is evident that the spinning motion of the ball in Fig. 137 causes the nose of the ball to move

upwards (towards the top of the page) and the ball in Fig. 137 is deflected upwards. *A spinning ball is always deflected in the direction in which its nose moves.\**

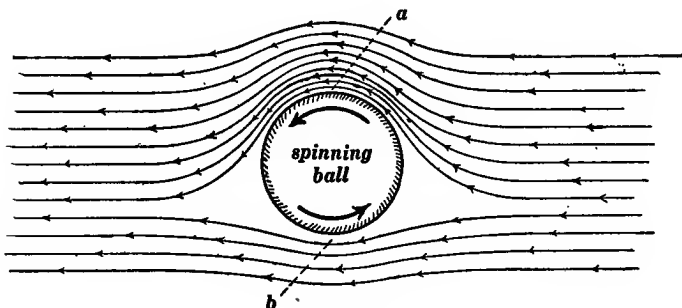


Fig. 136.

Blast of air blowing past a spinning ball.

**76. Quantitative examples of Bernoulli's principle.** (a) *Efflux of liquid from a tank.*—Liquid of density  $d$  issues as a jet through

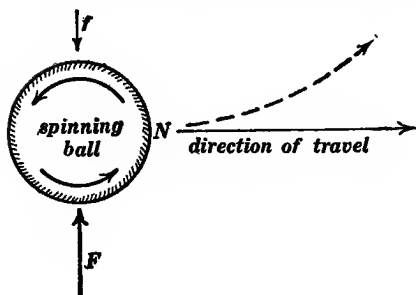


Fig. 137.

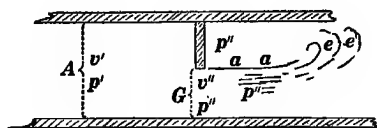


Fig. 138.

a hole in the side of a tank in which the pressure is  $p$ . In the tank, where the velocity of the liquid is inappreciable the total energy per unit volume of the liquid is  $p$  according to equation (i) of Art. 73. In the jet where the velocity of the liquid is  $v$

\* Bernoulli's principle is true only when friction is negligible and when the fluid motion is irrotational. But the air motion in Fig. 135 is due to friction between the air and the spinning ball, and there is only one particular case in which the air whirl in Fig. 135 is irrotational, namely, when the velocity of the air at a point is inversely proportional to the distance of the point from the axis of the spinning ball. Therefore the application of Bernoulli's principle to the air motion in Fig. 136 is very questionable.

the total energy per unit volume of the liquid is  $p' + \frac{1}{2}dv^2$ , where  $p'$  is the outside pressure. Therefore  $p = p' + \frac{1}{2}dv^2$  or  $p - p' = \frac{1}{2}dv^2$ .

(b) Figure 138 represents a partly opened gate valve. At  $A$  the sectional area of the pipe is  $A$ , the velocity of the fluid in the pipe is  $v'$  and its pressure is  $p'$ . The sectional area of the gate opening is  $G$ , and at  $G$  the velocity is  $v''$  and the pressure is  $p''$ . The pressure of the fluid drops from  $p'$  to  $p''$  as it enters the gate where

$$p' + \frac{1}{2}dv'^2 = p'' + \frac{1}{2}dv''^2 \quad (i)$$

and where

$$Av' = Gv'' \quad (ii)$$

but the pressure does not rise again beyond the gate because of the vortex sheet  $aa$  and the eddies at  $ee$ . *There is a loss of pressure through the gate and this loss of pressure is equal to  $p' - p''$ .*

(c) *The Venturi water meter.*—Let  $A$  be the sectional area of the pipe in Fig. 131 at  $A$  and let  $B$  be the sectional area of the pipe at  $B$ . Then, according to equation (17) of Art. 71, we have

$$Av' = Bv'' \quad (i)$$

and, according to equation (18) of Art. 74, we have

$$p' + \frac{1}{2}dv'^2 = p'' + \frac{1}{2}dv''^2 \quad (ii)$$

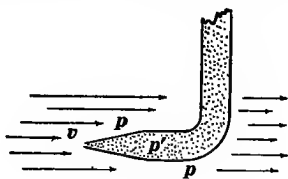
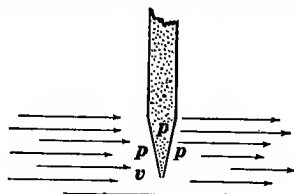
Now  $Av'$  is the discharge rate of the stream (cubic feet per second or cubic centimeters per second), and by using equations (i) and (ii)  $Av'$  can be expressed in terms of  $(p' - p'')$ ,  $d$ ,  $A$  and  $B$  so that if  $d$ ,  $A$  and  $B$  are known and if  $p' - p''$  is measured by a suitable pressure gauge, the value of  $Av'$  can be calculated.

(d) *The Pitot velocity meter.* Consider a stream of water flowing past a pointed tube as shown in Fig. 139a. The total energy per unit volume of the moving water is  $p + \frac{1}{2}dv^2$ , and the total energy per unit volume of the still water in the tube

is  $p'$ . Therefore, according to Bernoulli's principle,\* we have:

$$p + \frac{1}{2}dv^2 = p' \text{ whence } v = \sqrt{\frac{2(p' - p)}{d}} \quad (19)$$

Consider the pointed tube which is shown in Fig. 139*b*. Bernoulli's principle does not apply to the motion of the water near

Fig. 139*a*.Fig. 139*b*.

the tip of this tube. In fact the moving water is separated from the still water in the tube by an approximately flat vortex sheet and the pressure  $p$  of the still water in the tube is approximately equal to the pressure  $p$  of the moving water.

Therefore if two pointed tubes like Figs. 139*a* and 139*b* are arranged with their tips near together in a stream of fluid, the pressure in one tube will exceed the pressure in the other by the amount  $p' - p$ , and if this pressure difference is measured the

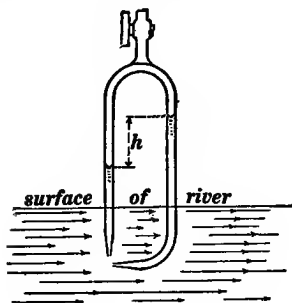


Fig. 140.

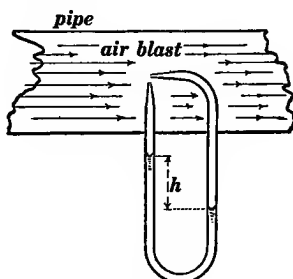


Fig. 141.

velocity  $v$  of the fluid can be calculated from equation (19), the

\* There is some doubt as to the applicability of Bernoulli's principle to the liquid in the neighborhood of the tip of the tube in Fig. 139*a* but experiment justifies it because equation (19) is verified (approximately) by experiment.

density  $d$  of the fluid being known. A pair of tubes arranged as stated and combined with a device for measuring  $p' - p$  is called a *Pitot meter* from its inventor.

Figure 140 shows a Pitot meter arranged for measuring the velocity of the water in a river. The pressure difference ( $p' - p$ ) of equation (19) is measured by the difference of level  $h$ , in fact  $p' - p = h d g$  according to equation (16) of Art. 62.

Figure 141 shows a Pitot meter arranged for measuring the velocity of an air blast in a pipe. The pressure difference  $p' - p$  is equal to  $h d' g$  where  $d'$  is the density of the liquid in the tube in Fig. 141. Of course  $d$  in equation (19) is in this case the density of the air in the blast.

#### PROBLEMS.

131. Calculate the velocity of efflux of kerosene from a tank in which the pressure is 50 "pounds" per square inch above atmospheric pressure, the density of kerosene being, say, 48 pounds per cubic foot.

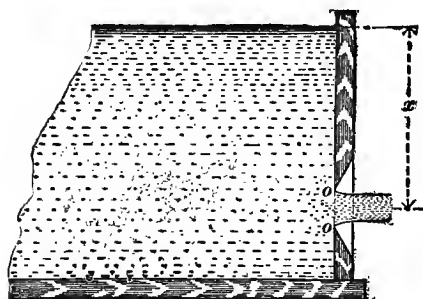


Fig. 142.

132a. Show that the velocity of efflux of water from the side of a tank as indicated in Fig. 142 is  $v = \sqrt{2gx}$  where  $g$  is the acceleration of gravity.

132b. How much water flows in one minute out of a hole one foot in diameter, the pressure in the tank being that due to a

head of water 16.1 feet above the center of the hole. Ans. 15.36 cubic feet per second.

*Note.*—Experiment shows that the velocity of efflux of water through a hole of this kind is about 0.98 of the theoretical velocity as calculated on the assumption that the water is frictionless.

The sectional area of the stream at a short distance from the hole (where the flow is approximately lamellar) is about 0.62 of the area of the hole, as found by experiment.

133. Water flows in a 12-inch main at a velocity of 4 feet per second and encounters a partly closed gate valve through which the stream of water is reduced to 0.36 square foot section. Calculate the loss of pressure at the valve due to friction.

134. A street water-main 7 inches inside diameter has in it a throat 3 inches inside diameter. Water flows through the pipe at the rate of 1.5 cubic feet per second and the pressure of the water in the 7-inch pipe is 90 "pounds" per square inch. What is the pressure in the throat, ignoring friction?

135. Calculate the rate of discharge of the stream which flows through the pipe in Fig. 131 when  $(p' - p'')$  is balanced by a mercury column  $6\frac{1}{2}$  inches high (density of mercury 13.6 times the density of water); the sectional area of pipe at  $A$  being 1.2 square feet, the sectional area of the pipe at  $B$  being 0.92 square feet and the density of the water in the stream being  $62\frac{1}{2}$  pounds per cubic foot.

*Note.*—A  $U$ -tube attached at  $A$  and  $B$  in Fig. 131 is water filled, with mercury in the lower part of the  $U$ , and the mercury level in one arm of the  $U$  is  $6\frac{1}{2}$  inches higher than in the other arm.

136. A fire-test is made to determine the friction loss of pressure in a street main when a fire hose is in action, and it is observed that the pressure at the hydrant drops from 150 "pounds" per square inch to 90 "pounds" per square inch when 6 cubic feet of water per second is being discharged, the sectional area of the pipe at the point of attachment of the pressure gauge being 0.0833 square foot. What is the friction loss of pressure between the reservoir and the point of attachment of the pressure gauge?  
Ans. 24.8 "pounds" per square inch.

*Note.*—It might seem that the required friction loss of pressure is 150 "pounds" per square inch *minus* 90 "pounds" per square inch. But this total loss of pressure, amounting to 60 "pounds" per square inch, has been partly used to set the water moving at the specified velocity (six cubic feet per second divided by 0.0833 square feet). It is assumed that the pressure gauge indicates the actual pressure in the pipe to which it is connected.

137. A pair of Pitot tubes is placed in a river as shown in Fig.

140, and the difference of level  $h$  is observed to be 3 inches. What is the velocity of the water in the river at the place where the meter is placed?

138. A pair of Pitot tubes is placed in an air blast as shown in Fig. 141, and the difference of level  $h$  is observed to be  $2\frac{1}{2}$  inches, the liquid in the tube being water. What is the velocity of the air blast?

*Note.*—In this problem ignore the compressibility of the air. Take the density of the air to be 0.0013 of the density of water.



**77. Fluid friction.\***—There are two fairly distinct kinds of friction which cause a loss of pressure as a fluid flows through a pipe or channel, and, although these two kinds of friction always exist together, there are two extreme cases in which each exists by itself, or nearly so.

**Viscous friction.**—When a fluid flows through a very small smooth-bore pipe, like a small glass tube, the friction is due almost wholly to the sliding of each layer of fluid over the adjoining layer, *and the volume of fluid flowing per second through such a tube is proportional to the pressure-difference between the ends of the tube.*

**Eddy friction.**—When a fluid flows through a large pipe or channel, the friction is due almost wholly to the formation of eddies. In this case *the volume of fluid flowing per second through the pipe is approximately proportional to the square root of the pressure-difference between the ends of the pipe.*

**Practical formula for calculating the frictional loss of pressure due to the flow of water or gas through a pipe.**—The formula which is used in practice for calculating the frictional loss of pressure in a pipe is only approximately true and therefore the formula has no rigorous derivation. The only thing to be done

\* No attempt is here made to discuss the fluid friction which opposes the motion of a boat.

in connection with it is to exhibit its meaning clearly, which is the purpose of the following argument. The flow of a fluid over a surface, such as the interior walls of a pipe, is opposed by a force which is approximately proportional to the *area* of the surface, to the *density* of the fluid and to the *square of the velocity* at which the fluid is flowing. Therefore, we may write

$$F = kadv^2 \quad (i)$$

in which  $a$  is the area of the surface,  $d$  is the density of the fluid,  $v$  is the velocity of flow, and  $F$  is the opposing force. The quantity  $k$  is sometimes called the *coefficient of friction* of the moving fluid against the walls of the pipe. It depends greatly upon the degree of roughness of the walls.

Consider a pipe of which the length is  $L$  and of which the inside diameter is  $D$ . The total area of interior walls of this pipe is  $\pi DL$ , so that, using  $\pi DL$  for  $a$  in equation (i), we have  $F = k\pi DLdv^2$  for the total opposing force acting on a fluid of density  $d$  which flows through the pipe at velocity  $v$ . This opposing force is equal to the difference of pressure at the two ends of the pipe multiplied by the sectional area  $\left(\frac{\pi D^2}{4}\right)$  of the bore of the pipe. Therefore, using  $p$  for the loss of pressure due to friction, we have

$$F = p \cdot \frac{\pi D^2}{4} = k\pi DLdv^2 \quad (ii)$$

whence

$$p = \frac{4kLdv^2}{D} \quad (20)$$

It is usually convenient to express the loss of pressure in terms of the volume  $V$  of fluid discharged per second instead of expressing it in terms of the velocity  $v$  of the fluid. According

to Art. 71  $V = \pi r^2 v = \frac{\pi}{4} \cdot D^2 v$ , so that  $v = \frac{4V}{\pi D^2}$ . Therefore, sub-

stituting this value for  $v$  in equation (20). we get

$$p = \frac{64kLdV^2}{\pi^2 D^5} \quad (21)$$

If the loss of pressure  $p$  is expressed in "pounds" per square foot, the length of the pipe  $L$  in feet, the diameter  $D$  of the pipe in feet, the density  $d$  of the fluid in slugs per cubic foot, and the discharge rate  $V$  in cubic feet per second, then the value of  $k$  is about 0.00264 for water in ordinary cast iron pipes and about 0.00179 for gas or air in cast iron pipes.

**Example.**—It is required to find the inside diameter  $D$  of a cast iron pipe to bring 10 cubic feet of water per second from a reservoir to a distributing point in a city, the length of the pipe being 16,000 feet and the allowable loss of pressure being that which corresponds to a head of 200 feet of water. That is, if the reservoir is 350 feet above the distributing point, then there is to be an available head of 150 feet at the distributing point.

The pressure corresponding to 200 feet head may be calculated by using equation (16) of Art. 62, and it is very nearly 12500 "pounds" per square foot. Using this for  $p$  in equation (21), using  $k = 0.00264$ , using  $d = 1.94$  slugs per cubic foot, using  $L = 16,000$  feet, and using 10 cubic feet per second for  $V$  we get  $D = 1.336$  feet as the required diameter of pipe.

#### PROBLEMS.

**139a.** How many gallons of water per hour will flow through 1000 feet of one-inch pipe with an available head of 10 feet (available pressure of 625 "pounds" per square foot) to overcome friction? One gallon = 231 cubic inches.

**139b.** Calculate the discharge of water in cubic feet per second through a pipe 10 feet long and 2 inches inside diameter, available head being 10 feet (available pressure 625 "pounds" per square foot). Make allowance for friction and also for the fact that

part of the available head (or pressure) is used to set the water in motion.

**140.** Find the diameter  $D$  of the pipe required to deliver 5,000,000 gallons of water per day with a friction loss of pressure amounting to 100 "pounds" per square inch, length of pipe line 7 miles.

**141a.** A 2-mile gas main in a city delivers daily 36,000 cubic feet of gas (density about 0.0008 of the density of water), and the friction loss of pressure in the pipe is 0.5 "pound" per square inch. What is the diameter of the pipe?

**141b.** The rim of a fly wheel is 30 feet in circumference and one foot wide, and it has a velocity of 75 feet per second. The rim is surrounded by a band of metal with a uniform, narrow air space between. Calculate the friction drag on the rim of the flywheel by the air between the rim and the band. How much power is lost because of this friction?

*Note.* The windage friction of a fly wheel is greatly decreased by enclosing the wheel in a smooth casing because of the great reduction of fan action brought about thereby. The relative velocity of fly-wheel rim and air-layer in this problem is about 37.5 feet per second. Use equation (i) on page 131, taking 0.00179 for  $k$ .

## CHAPTER VI.

### THE STATICS OF ELASTICITY.

**78. Stress and strain.**—When a fluid is compressed, or when a wire is stretched or twisted, or when a beam is bent *the adjacent parts of the fluid or wire or beam push or pull on each other*. This force action between contiguous parts of a distorted body is called *stress*.\*

When a fluid is compressed, or when a wire is stretched or twisted or when a beam is bent the change of size or shape is called *strain*, and in many cases one is obliged to consider primarily, not the change of size or shape of the body as a whole, but the change of size or shape of a small part of the body.

**Homogeneous and non-homogeneous strains and stresses.**—When a wire is stretched or when a column is shortened under a moderate load every portion of the wire or column is similarly distorted and the internal force actions are alike everywhere in the stretched wire or shortened column; and we have what is called *homogeneous strain* and *homogeneous stress*. When the hydrostatic pressure is the same throughout a body of fluid we have an example of *homogeneous stress*, and if the body of fluid is everywhere of the same kind each portion of the fluid will be compressed (reduced in volume) to the same extent; and we will have a *homogeneous strain*.

In a twisted wire or bent beam, however, the distortion (strain) is not the same everywhere nor is the internal force action (stress) the same everywhere, and we have what is called *non-homogeneous strain* and *non-homogeneous stress*. Consider for example a portion of the material near the axis of a twisted

\* The *outside forces* which act on a compressed fluid, or on a stretched or twisted wire, or on a bent beam are sometimes called stresses, but the word stress is here understood to refer to the force actions inside of a distorted body, to the force actions between contiguous parts of a body, as stated.

wire; this portion is certainly not distorted to the same extent as a portion of the material near the outside of the wire.

**79. Longitudinal stress and strain.**—The stress and strain in a stretched wire is called *longitudinal stress* and *longitudinal strain*. Let  $F$  be the total stretching force acting on a wire and let  $q$  be the sectional area of the wire, then *the stretching force per unit of sectional area*,  $F/q$ , is the expression for the stress. Let  $\Delta l$  be the elongation of the wire and let  $l$  be the initial length of the wire, then *the elongation per unit of initial length*,  $\Delta l/l$ , is the expression for the strain. These statements all apply to a loaded column except that in case of a loaded column  $F$  and  $\Delta l$  are both considered as negative.

**Definition of stretch modulus.**—The strain  $\Delta l/l$  in a stretched wire or loaded column is proportional to the stress  $F/q$  if the stress is not carried beyond what is called the *elastic limit* (Hooke's law). Therefore stress divided by strain is a constant for the given material, that is

$$E = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{Fl}{q \cdot \Delta l} \quad (22)$$

where  $E$  is a constant which is called the *stretch modulus*\* of the substance of which the stretched wire or loaded column is made. It is evident from this equation that the stretch modulus is expressed in terms of force per unit area inasmuch as the units of length in  $\Delta l$  and  $l$  cancel. Thus the stretch modulus is expressed in dynes per square centimeter (in the c.g.s. system). or in "pounds" per square foot (in the f.s.s. system). In the accompanying table the stretch modulus is expressed in "pounds" per square inch.

TABLE.

STRETCH MODULUS IN "POUNDS" PER SQUARE INCH.

Copper wire.....	17,700,000
Steel (rolled).....	29,800,000
Wrought iron.....	29,600,000
Cast iron.....	16,000,000
Glass (ordinary window glass).....	9,600,000
Oak wood (with grain).....	1,450,000
Poplar wood (with grain).....	750,000

\* Frequently called Young's modulus.

**80. Definition of bulk modulus.**—Consider a substance whose volume is  $v$  at pressure  $p$ , and whose volume is decreased by the amount  $\Delta v$  when the pressure is increased by the amount  $\Delta p$ . In this case the strain is expressed as  $\Delta v/v$ , the stress is expressed as  $\Delta p$ , and the ratio stress/strain is a constant which is called the *bulk modulus*  $B$  of the substance. That is

$$B = \frac{v \cdot \Delta p}{\Delta v} \quad (23)$$

For liquids and for solids like glass and steel the strain  $\Delta v/v$  is very nearly proportional to  $\Delta p$  when  $\Delta p$  has any value up to many thousands of "pounds" per square inch, but for gases  $\Delta v/v$  is proportional to  $\Delta p$  for very small increases of pressure, only. Thus for ordinary air initially at 15 "pounds" per square inch,  $\Delta v/v$  is about 0.06 when  $\Delta p$  is one "pound" per square inch, and  $\Delta v/v$  is about 0.4 when  $\Delta p$  is ten "pounds" per square inch.

**Remark.**—The reciprocal of the bulk modulus of a substance is called the *compressibility* of the substance.

#### TABLE.

COMPRESSIBILITIES AT 20° C. FOR MODERATE INCREASE OF PRESSURE.

(Decrease of volume per unit initial volume when pressure is increased one "pound" per square inch)

Ether.....	0.000,000,115
Alcohol.....	0.000,000,069
Water.....	0.000,000,031,3
Glass.....	0.000,000,001,5
Steel.....	0.000,000,000,46

**81. ELASTIC HYSTERESIS.**—In nearly all substances there is more or less of a tendency for strain to persist after the stress has ceased. This is of course very markedly the case when a substance is strained beyond the elastic limit, but we refer here to the persistence of strains far below the elastic limit. When a wire, for example, is subjected to a stress (a tension) which increases and decreases repeatedly between two given values  $S_1$  and  $S_2$ , the relation between stress and strain is somewhat as indicated in Fig. 143, where ordinates represent stress and abscissas represent strain. Branch  $a$  of the curve represents the relation between stress and strain while the stress is increasing, and branch  $b$  represents the relation between stress and strain while stress is decreasing. The divergence of the two

branches *a* and *b* is due to what is called *elastic hysteresis*. Stress is here supposed to increase and decrease very slowly; if the stress increases and decreases rapidly the divergence of the two branches is due not only to what is here called hysteresis but also to *elastic lag*.

82. ELASTIC LAG AND VISCOSITY. Many substances, glass for example, when suddenly subjected to stress take on a certain amount of strain quickly after which the strain increases for a long time; and when a stress is suddenly relieved a remnant of the strain persists for a long time. This phenomenon is called *elastic lag*.

In a substance like pitch the distortion (strain) continues indefinitely when the substance is subjected to stress (except the stress be a hydrostatic pressure). Such substances are said to be *viscous*. Nearly all metals are more or less viscous under excessive stress.

83. ELASTIC FATIGUE. The repeated application of a stress weakens a metal so that it will break at less than its normal breaking stress, at a stress which is less,

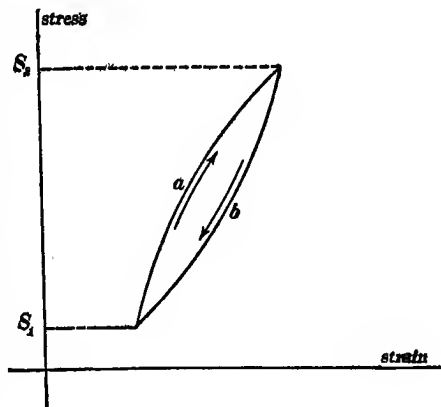


Fig. 143.

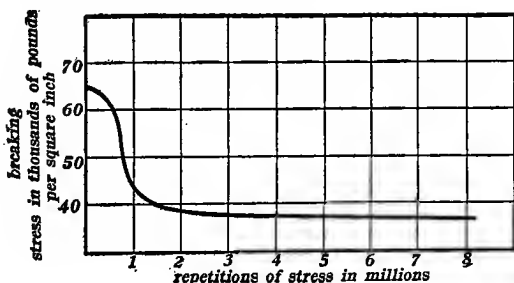


Fig. 144.

even, than that which corresponds to its elastic limit. Thus Fig. 144 shows the decrease of breaking stress (longitudinal) for a sample of mild steel with repetitions of the stress.

#### PROBLEMS.

142. A steel wire 0.05 inch in diameter and 10 feet initial length is elongated 0.060 inches by a stretching force of 29 pounds.

Find value of stress, find value of strain, and find value of stretch modulus.

143. What elongation will be produced when 1000 feet of copper wire 0.08 inch in diameter is subjected to a stretching force of 10 "pounds"?

144. A steel rod is subjected to increasing tension, and the stresses and strains are plotted in Fig. 145. Curve *B* is the same as curve *A* except for change of scale as indicated in the figure. What is the stretch modulus of the steel? What is the approximate elastic limit of the steel?

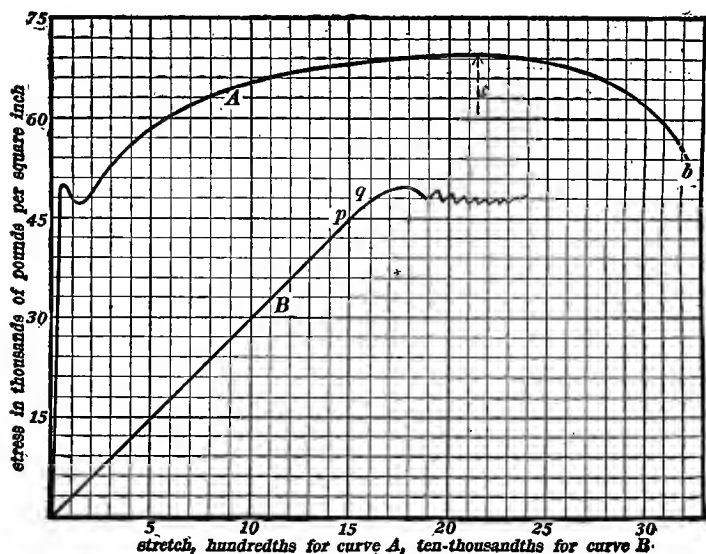


Fig. 145

145a. The density of sea water is 64.9 pounds per cubic foot at the surface of the sea. What is its density at 4000 fathoms where the pressure is about 1,560,000 "pounds" per square inch?

145b. A thick-walled steel cylinder contains a 12-inch column of alcohol under a one-inch piston. Find the movement of the piston, when it is acted upon by a force of 500 pounds, neglecting the expansion of the cylinder.

146. The relation between the pressure  $p$  and the volume  $v$  of a gas (at constant temperature) is  $pv = \text{a constant}$  (Boyle's law). Find decrement of volume  $\Delta v$  due to increase of pressure  $\Delta p$  and show that the bulk modulus of a gas is numerically equal to the pressure  $p$  of the gas.

**Remark.**—When a gas is quickly compressed its temperature rises and the increase of pressure due to a given decrease of volume is greater than it would be if the temperature were constant. Consequently the bulk modulus of a gas is greater for quick compression than it is for very slow compression (constant temperature compression).

84. **The bent beam.**—When a beam is bent the filaments on one side of the beam are lengthened and the filaments on the

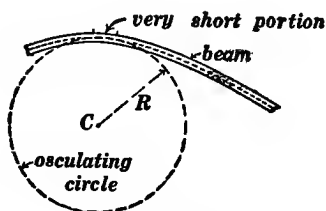


Fig. 146.

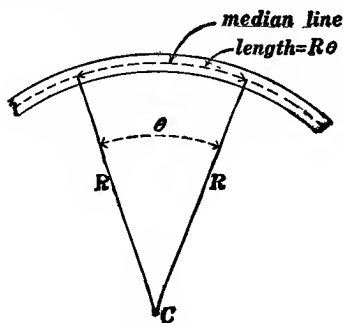


Fig. 147

other side of the beam are shortened, and the amount of this lengthening or shortening depends, among other things, upon the degree of curvature of the bent beam; but the degree of curvature of a bent beam usually varies from point to point along the beam, and therefore if we wish to determine the extent of lengthening and shortening it is necessary, in general, to consider a very short portion of the beam. A very short portion of a beam has the same curvature as the osculating circle as shown in Fig. 146, and the easiest way to think of a very short

portion of the bent beam is to think of it as a portion of a long beam which is actually bent into the arc of a circle, as indicated in Fig. 147.

Figure 148 represents a portion of the beam which is shown in

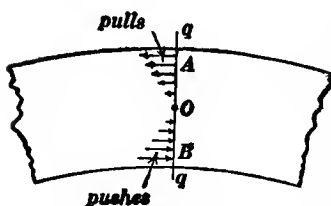


Fig. 148.

Fig. 147. The total force action exerted across the section  $qq$  of the beam (force action exerted **BY** the material on one side of  $qq$  **ON** the material on the other side of  $qq$ ) is a turning force or torque  $T$  acting about the axis  $O$  at right angles to the plane of the paper.

Let us consider the portion of the beam which lies between the two radii  $RR$  in Fig. 147. This portion of the beam is shown somewhat enlarged in Fig. 149. The original length of

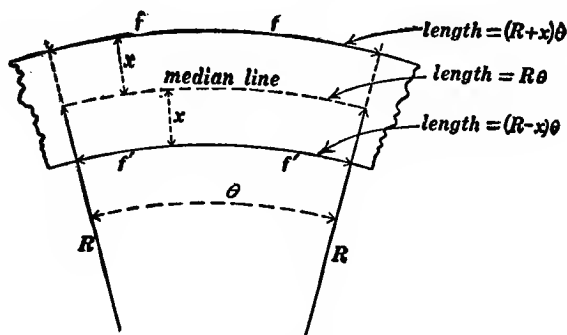


Fig. 149.

every filament of this portion of the beam is  $R\theta$ . Consider the upper filament  $ff$  of the beam. The stretched length of this filament is  $(R+x)\theta$ , because  $ff$  is a circular arc of radius  $R+x$  which subtends the angle  $\theta$ . Therefore the filament  $ff$  has been increased in length by the amount  $x\theta$  by the bending. Similarly it may be shown that the lower filament  $f'f'$  has been shortened by the amount  $x\theta$  by the bending.

Let us imagine the beam to grow by the addition of layers of material of thickness  $\Delta x$  on top and bottom as indicated in the

sectional view, Fig. 150, and let us find the corresponding increase  $\Delta T$  of the torque  $T$  (see above). According to equation (22) the actual tension (or pull) in the upper layer is

$$F = \frac{E \times b \cdot \Delta x \times x \theta}{R \theta}$$

because the sectional area of the upper layer is  $q = b \cdot \Delta x$ , the increase of length of the filaments in this layer is  $\Delta l = x \theta$  and the initial length of the filaments in this layer is  $l = R \theta$ , as

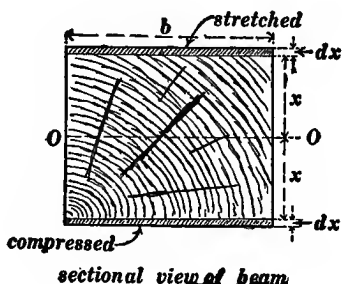


Fig. 150.

above explained,  $E$  being the stretch modulus of the material of which the beam is made. Therefore  $F = \frac{Eb}{R} x \cdot \Delta x$ . Now this stretching force is a pull at  $A$  in Fig. 148 and its torque action about  $O$  is  $Fx$  which is equal to  $\frac{Eb}{R} x^2 \cdot \Delta x$ . Considering the compression in the lower layer in Fig. 150 as a negative tension we may apply equation (22) as before and get  $\frac{Eb}{R} x^2 \cdot \Delta x$  as the torque action about  $O$  due to the push at  $B$ , Fig. 148, in the compressed added layer. These two torque actions are in the same direction about  $O$ , and therefore

$$\Delta T = \frac{2Eb}{R} x^2 \cdot \Delta x$$

or

$$\frac{dT}{dx} = \frac{2Eb}{R} x^2$$

Now if  $y = ax^3$  it is easy to show that  $\frac{dy}{dx} = 3ax^2$  and, therefore, if  $3a = \frac{2Eb}{R}$ , or if  $a = \frac{2Eb}{3R}$ , we have  $y = \frac{2Eb}{3R} x^3$  whose

derivative is equal to  $\frac{dT}{dx}$  so that

$$T = y + C = \frac{2Eb}{3R}x^3 + C$$

But, evidently  $T = 0$  if  $x = 0$ , that is an infinitely thin beam requires no torque to bend it, so that the constant  $C$  must be zero. Furthermore let  $d$  be the total depth of the beam ( $= 2x$ ) then  $x = d/2$  and we get

$$T = \frac{1}{12} \frac{E}{R} bd^3 \quad (24)$$

This equation applies to a beam bent in any manner whatever,\*  $T$  being the torque action exerted across a section of the beam at a place where the radius of curvature of the beam is  $R$ .

#### PROBLEMS.

**147.** The elastic limit of mild steel corresponds to a stretch of, say, 1.4 parts in 1000. Find the radius of curvature (of middle line of the beam) of the sharpest bend that can be given to the beam in order that the greatest strain in the beam may not exceed the elastic limit; depth of beam  $d$  being 3 inches, and breadth  $b$  2 inches. What is the greatest tension in the beam in "pounds" per square inch? What is the greatest compression in "pounds" per square inch? What torque action is exerted across a section of the beam?

**148.** A beam  $\frac{1}{2}$  inch wide and  $\frac{1}{2}$  inch deep is carried by two supports  $SS$  as shown in Fig. 151. The two forces  $F$  are each 32 pounds, the distances  $L$  are each 12 inches, the distance  $L'$  is 36 inches, and the center of the beam moves 0.4 inch upwards when the forces  $F$  are applied. Find the

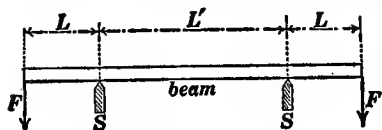


Fig. 151.

stretch modulus of the beam material.

\* Bent in a plane which is parallel to the dimension  $d$ .

*Note.*—The portion of the beam between the two supports is everywhere acted upon by the same bending torque (bending moment, as it is usually called) and therefore this portion of the bent beam forms the arc of a circle.

**85. Shearing stress and shearing strain.**—The type of stress and strain in a twisted wire or rod is called *shearing stress* and *shearing strain*, and the discussion of this type of stress and strain is somewhat difficult chiefly because there is no familiar example of *homogeneous* shearing stress and strain. The stress and strain in a twisted rod are *non-homogeneous*. Any intelligible discussion of shearing stress and strain must, however, be based upon on the consideration of homogeneous shear.

**Homogeneous shearing stress and strain.**—Figure 152 represents a cubical portion  $ABCD$  of a substance. Outward forces ( $S$  units of force per unit area) act on faces  $AB$  and  $CD$ , inward forces ( $S$  units of force per unit area) act on faces  $AC$  and  $BD$ , and no force at all acts on the faces of the cube which are parallel to the plane of the paper.

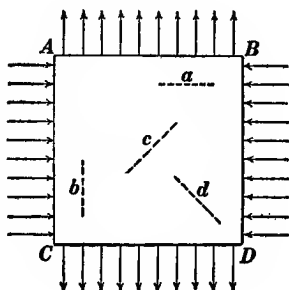


Fig. 152.

There is a pull across section  $a$ , or across any section parallel to  $a$ ; there is a push across section  $b$ , or across any section parallel to  $b$ ; and there is a tangential stress (tendency to slide) across sections  $c$  and  $d$  or across any section parallel to  $c$  or  $d$ . The stress (pull, or push, or tendency to slide) is in each case  $S$  units of force per unit area of section. This is evident in case of sections  $a$  and  $b$ , and it is easily proven for sections  $c$  and  $d$ .

The forces which are shown in Fig. 152 *shorten* the cube in one direction and *lengthen* it in the other direction as indicated by the dotted rectangle in Fig. 153, and the value of the distortion (strain) is always specified by giving the value of the angle ( $90^\circ - \theta$ ) in radians. This angle, which is represented by  $\phi$ , is called the *angle of shear*. The angle of shear is most clearly

shown in Fig. 154 in which a cube  $efgh$  is so distorted that the face  $efgh$  becomes a rhombus as indicated by the dotted lines.

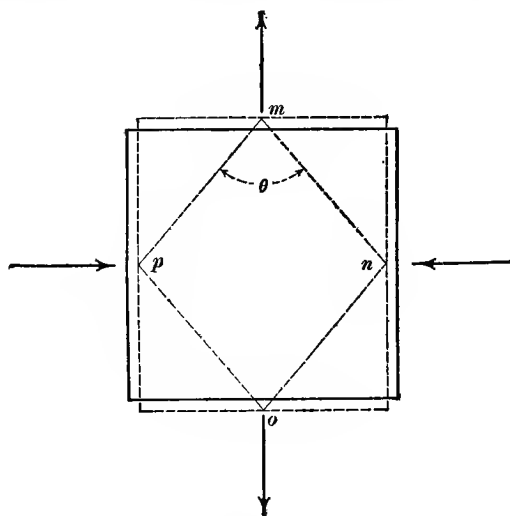


Fig. 153.

The angle  $\phi$  is always very small and it is evidently equal to  $l/L$ , where  $L$  is the length  $eg$ .

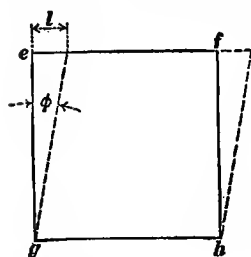


Fig. 154.

**Slide modulus of a substance.**—The angle of shear is proportional to the shearing stress  $S$  (Hooke's law). Therefore we have

$$\frac{S}{\phi} = n \quad (25)$$

where  $n$  is a constant which is called the *slide modulus*\* of the material.

**Discussion of a twisted rod.**—Consider a cylindrical rod of radius  $r$  and length  $L$ , and suppose that one end of the rod is fixed while the other end is turned through the angle  $\alpha$ . The

\* Often called the *coefficient of simple rigidity*.

torque  $T$  which is required to twist the rod is transmitted across each section of the rod, and if we wish to derive an expression for  $T$  we must as a first step find the increment  $\Delta T$  due to an imagined growth  $\Delta r$  of the radius of the rod. Therefore let us consider a layer or shell at the surface of the rod, thickness of this layer or shell being  $\Delta r$ .

This layer or shell is shown *developed* (unrolled so as to be flat) in Fig. 155, in which  $AB$  represents the developed shell *before* the rod is twisted, and  $A'B'$  represents the developed shell *after* the rod is twisted. It is evident from Fig. 155 that every portion of the shell is under a shearing strain, and of course the stress

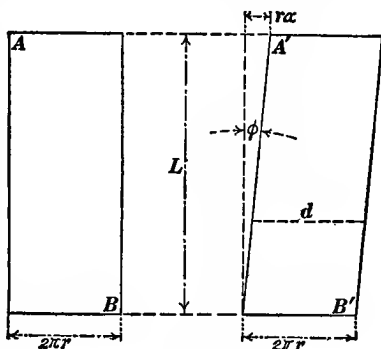


Fig. 155.

in the material of the shell is a shearing stress. The angle of shear is

$$\phi = \frac{r\alpha}{L} \quad (i)$$

Consider the section  $d$  of the shell. This section transmits the torque  $\Delta T$ , and to find an expression for  $\Delta T$  it is to be noted, 1st, that the force action across  $d$  is a tangential stress  $S$  units of force per unit area where  $S = n\phi$  according to equation (25), 2d, that the area of the section  $d$  is  $2\pi r.\Delta r$ , and, 3d, that the force acting across each portion of  $d$  is at right angles to the radius  $r$  so that the torque due to each portion of the force is found by multiplying the force by  $r$ . Therefore

$$\Delta T = 2\pi r.\Delta r \times n\phi \times r,$$

but  $\phi = \frac{r\alpha}{L}$ , and therefore we get

$$\frac{dT}{dr} = \frac{2\pi n\alpha.r^3}{L} \quad (ii)$$

It is easy to show that  $y = \frac{\pi n \alpha . r^4}{2L}$  is a function whose derivative  $\frac{dy}{dr}$  is equal to  $\frac{dT}{dr}$ . Therefore

$$T = \frac{\pi n \alpha . r^4}{2L} + a \text{ constant} \quad (\text{iii})$$

But  $T = 0$  when  $r = 0$ ; because no torque at all is required to twist a wire of zero radius. Therefore the constant of integration is zero, and equation (iii) becomes

$$T = \frac{\pi n \alpha . r^4}{2L} \quad (26)$$

#### PROBLEMS.

**149.** Prove that the force action across section  $c$  or section  $d$  in Fig. 152 is purely tangential (tendency to slide) and equal to  $S$  units of force per unit area of section.

*Note.*—The stress in Fig. 152 being homogeneous the force action on any section parallel to  $c$  (or  $d$ ) is the same as on section  $c$ . Therefore consider the diagonal plane  $CB$  (perpendicular to the plane of the paper in Fig. 152) and find the *magnitude, direction and line of action* of the force which must act across this diagonal plane on the half  $ABC$  of the cube to balance the forces which act on faces  $AB$  and  $AC$ .

**150.** A torque of 41 “pound” feet applied to one end of a steel shaft which is one inch in diameter and 12 feet long, turns the end through 2.86 degrees. Find the slide modulus of the steel.

**151.** A 500-gram metal disk 10 centimeters in radius is hung with its axis of figure coincident with a steel suspending wire. The radius of the wire is 0.5 millimeter and the length of the wire is 2 meters. Find the number of rotational oscillations which the disk will make in 5 minutes. The slide modulus of steel is  $8.3 \times 10^{11}$  dynes per square centimeter.

*Note.*—The coefficient of  $\alpha$  in equation (26) is the “stiffness coefficient” which is mentioned in Art. 57.

## CHAPTER VII.

### THE DYNAMICS OF ELASTICITY.

**86. Oscillatory motion and wave motion.**—A very simple aspect of the dynamics of elasticity is discussed in Arts. 43-46 of Chapter II and in Art. 57 of chapter III. The discussion there given refers to what is sometimes called *concentrated elasticity* and *concentrated mass*, because the whole of the elasticity is assumed to be in one part of a system and the whole of the mass is assumed to be in another part of the system. Thus the mass of the moving spring in Fig. 86, chapter II, is assumed to be zero and the elasticity of the moving ball is assumed to be negligible. In the case of a stretched string, or an air column or a steel rod, however, the motion depends on the mass and elasticity of every part of the string or air column or rod. In the case of a string or air column or steel rod we have what is called *mixed elasticity and mass*, sometimes also called *distributed elasticity and mass*.

What is called *wave motion* always shows itself in a system where elasticity and mass are mixed (distributed), and it is the purpose in this chapter to discuss some of the simpler phases of wave motion. *Indeed this chapter considers only what are called wave pulses, not wave trains*, because, as Heaviside says, the physics of the wave train is very complex\* whereas the physics of the wave pulse is simple, physics in this connection meaning dynamics.

The mathematical theory of the dynamics of a system in which elasticity and mass are mixed is based upon the partial differential equation which expresses Newton's second law of motion as applied to an indefinitely small portion of such a system. If we

\* Let one consider such things as dispersion and the difference between group velocity and wave velocity, and the meaning of Heaviside's statement will be apparent.

leave friction and resistance out of account this equation is comparatively easy to handle (to integrate), and therefore friction and resistance are assumed to be zero throughout the following discussion.

This chapter is necessarily out of reach of the student who has not had a good course in calculus, and therefore no attempt is made to simplify the discussion as in the previous chapters.

**87. Equation of a traveling curve.** **Differential equation of travel.**—Let  $y = f(x')$  be the equation of a curve referred to axes which move along with the curve as it travels at uniform velocity  $v$  in the direction of the  $x$ -axis (towards the right), and let  $x = x' + vt$  or  $x' = x - vt$  where  $x'$  is the abscissa of the point  $p$ , Fig. 156, referred to the moving origin  $O'$ ,  $x$  is the

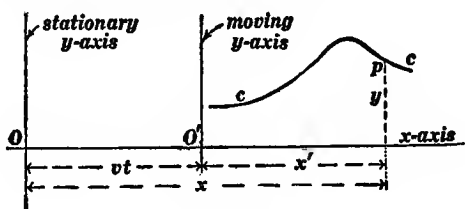


Fig. 156.

abscissa of  $p$  referred to the stationary origin  $O$ , and  $vt$  is the distance from the stationary origin  $O$  to the moving origin  $O'$ . Putting  $x - vt$  for  $x'$  in the equation  $y = f(x')$  we get

$$y = f(x - vt) \quad (i)$$

as the equation of the traveling curve  $cc$  referred to the fixed axes of reference.

Similarly,

$$y = F(x + vt) \quad (ii)$$

is the equation to a curve which travels to the left at velocity  $v$ . Differentiating equation (i) twice with respect to  $x$  and twice with respect to  $t$  we may eliminate the unknown function, and

the same may be said of equation (ii). In either case we get

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad (27)$$

which may very properly be called the differential equation of travel.

**88. The principle of superposition.** A property of linear differential equations.—A principle of extremely wide application in physics is the so-called *principle of superposition*. From the physical point of view a general statement of the principle is hardly possible, and therefore the following examples must suffice: (1) A person at *A* in Fig. 157 can see lamp No. 1 and another person at *B* can see lamp No. 2 *at the same time*. This means that the two beams of light *a* and *b* travel through the same region *R* at the same time without getting tangled up, as it were; each beam behaves as if it were traveling through the region alone.

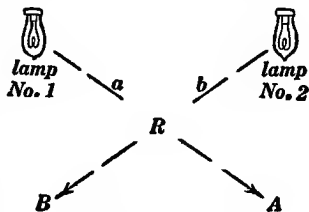


Fig. 157.

(2) Two sounds travel through the same body of air simultaneously, each sound traveling as if it were alone. (3) Two systems of water waves travel over the same part of a pond simultaneously, each system behaving as if the other system were non-existent. (4) Two messages\* can travel over a telegraph wire simultaneously and not get mixed up. (5) Two forces *F* and *G* exerted simultaneously on an elastic structure produce an effect which is the sum of the effects which would be produced by the forces separately, provided the strain of the structure is not carried beyond the elastic limit.

All of the effects in physics which are superposable—and this includes by far the greater portion of the effects in mechanics,

\* Indeed any number of distinct messages can travel over a telegraph wire in either direction or in both directions simultaneously. The only limiting feature in multiplex telegraphy or telephony is the design of the sending and receiving apparatus. In each of the above examples the word *two* means *two or more*.

heat, electricity and magnetism, light and sound, and chemistry—are expressible in terms of linear differential equations, and the principle of superposition is a clearly defined property of such equations as follows: *If  $y$  is a function of  $x$  which satisfies a linear differential equation, and if  $z$  is another function of  $x$  which satisfies the same differential equation, then  $(y + z)$  is a function of  $x$  which satisfies the differential equation.*

This proposition is true for both ordinary and partial linear differential equations, and although nearly all of the superposable effects in physics and chemistry are expressed in terms of partial differential equations, it is sufficient to prove the proposition for the ordinary linear differential equation because the proof is nearly the same for ordinary and for partial linear differential equations.

Given the linear differential equation

$$u + A \frac{du}{dx} + B \frac{d^2u}{dx^2} = 0 \quad (i)$$

Let  $y$  be a function which satisfies this equation. Then

$$y + A \frac{dy}{dx} + B \frac{d^2y}{dx^2} = 0 \quad (ii)$$

Let  $z$  be a function which satisfies this differential equation. Then

$$z + A \frac{dz}{dx} + B \frac{d^2z}{dx^2} = 0 \quad (iii)$$

Now

$$\frac{d(y + z)}{dx} = \frac{dy}{dx} + \frac{dz}{dx} \quad \text{and} \quad \frac{d^2(y + z)}{dx^2} = \frac{d^2y}{dx^2} + \frac{d^2z}{dx^2}$$

Therefore, adding equations (ii) and (iii) we get

$$(y + z) + A \frac{d(y + z)}{dx} + B \frac{d^2(y + z)}{dx^2} = 0$$

which, being identical to equation (i) in form, shows that  $(y + z)$  is a function which satisfies equation (i).

**89. Undetermined constants in the solution of ordinary differential equations.**—It is sufficient, perhaps, to illustrate by two simple examples.

(1) Consider the simple ordinary differential equation

$$\frac{dy}{dt} = a$$

where  $a$  is a constant. This equation means that the rate of change of  $y$  is constant and equal to  $a$ . Therefore the increase of  $y$  during time  $t$  is  $at$ , and the value of  $y$  at the instant  $t$  is

$$y = at + C$$

where  $C$  is the initial value of  $y$ , the value of  $y$  when  $t = 0$ .

(2) Consider the ordinary differential equation

$$\frac{d^2y}{dt^2} = a$$

where  $a$  is a constant. Then

$$\frac{dy}{dt} = at + B \quad \text{and} \quad y = \frac{1}{2}at^2 + Bt + C$$

where  $B$  is the value of  $\frac{dy}{dt}$  when  $t = 0$  and  $C$  is the value of  $y$  when  $t = 0$ .

**Undetermined functions in the solution of a partial differential equations.**—It is sufficient, perhaps; to illustrate by several simple examples.

(1) A boy saves money at the constant rate of \$5 per year, beginning at 14 years of age. Integrating with respect to  $b$  (the boy), we find that the boy has \$35 when he comes of age; but does he? Something surely depends on the boy's "old man," and we may represent by  $F(m)$ , a function of  $m$ , the money that the boy's "old man" has saved for him so that the amount of money the boy has when he comes of age will be \$35 +  $F(m)$ ;  $b$  and  $m$  are independent variables, let us say,

and a "constant of integration" with respect to  $b$  turns out to be a function of  $m$ .

(2) Concerning a hill it is known, only, that its slope,  $\frac{\partial y}{\partial x}$ , in the direction of the  $x$ -axis is constant and equal to  $a$ , so that

$$\frac{\partial y}{\partial x} = a$$

Before the complete hill or surface can be constructed from this differential equation an arbitrary

starting curve  $cc$ , Fig. 158, must be chosen. Let  $y = F(z)$  be the equation to this curve  $cc$ , then  $F(z)$  is the height of the hill above the point  $z$  on the  $z$ -axis, and  $ax + F(z)$  is the height of the hill above the point  $(x, z)$  in the  $xz$ -plane. That is

to say, the integration of  $\frac{\partial y}{\partial x} = a$

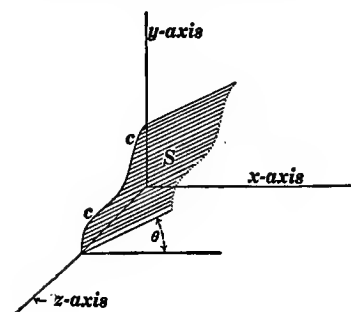


Fig. 158.

gives

$$y = ax + F(z)$$

**Proposition.**—The general solution of an ordinary differential equation of the  $n$ th order contains  $n$  unknown constants of integration. The general solution of a partial differential equation of the  $n$ th order contains  $n$  unknown functions of integration. No attempt is here made to prove this proposition in general.

**90. General solution of equation (27).**—Knowing that equations (i) and (ii) of Art. 87 are particular solutions of (27), we know, from the principle of superposition, that

$$y = F(x + vt) + f(x - vt) \quad (i)$$

is a solution of (27), and since it contains two unknown functions it is the general solution of (27).

91. **The differential equation of the transverse motion of a stretched string.**—When a stretched string is in equilibrium it is, of course, straight, gravity being neglected. Let us choose the equilibrium position of the string as the  $x$ -axis of reference. We will assume that each particle of the string moves only in a direction at right angles to the string (parallel to  $y$ -axis of reference), and that the string is perfectly flexible which means that the only forces to be considered are the forces due to tension of the string. An important consequence of the first assumption is that the  $x$ -component of the tension of the string has everywhere and always a constant value which we will represent by the letter  $T$  which is the tension of the string when it is in equilibrium.

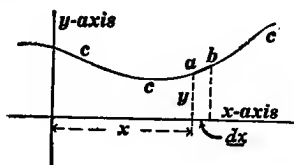


Fig. 159.

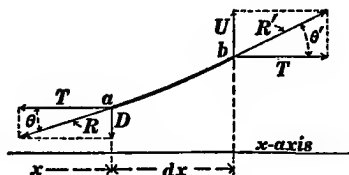


Fig. 160.

Let the curve  $ccc$  Fig. 159 be the configuration of the string at a certain instant  $t$ , that is to say,  $ccc$  is what a photographer would call a “snap shot” of the moving string at the instant  $t$ . The shape of the curve defines  $y$  as a function of  $x$ , and the steepness of the curve at any point whose abscissa is  $x$  is expressed by the derivative  $\frac{\partial y}{\partial x}$ .\*

Consider the very short portion  $ab$  of the string. The length of this portion when the string lies along the  $x$ -axis (in equilibrium) is  $dx$ , and its mass is  $m \cdot dx$ , where  $m$  is the mass per unit

\*A reader who fails to appreciate the fact that we are considering the state of affairs at a given instant, or that time is supposed to stop, as it were, may wonder why we use the notation of the partial derivative  $\frac{\partial y}{\partial x}$  instead of the more familiar notation  $\frac{dy}{dx}$ .

length, of the string. An enlarged view of the very short portion  $ab$  is shown in Fig. 160. The adjacent portions of the string pull on the portion  $ab$ . This pull at end  $a$  is represented by  $R$  and it is parallel to the string at  $a$ . This pull at  $b$  is represented by  $R'$  and it is parallel to the string at  $b$ . The  $x$ -component of  $R$  is the force  $T$  pulling towards the left and the  $x$ -component of  $R'$  is the force  $T$  pulling towards the right. Therefore the downward force  $D$  (see Fig. 160) is equal to  $T \tan \theta$ , the upward force  $U$  is equal to  $T \tan \theta'$ , and the net, unbalanced, upward force acting on the portion  $ab$  is

$$dF = U - D = T \tan \theta' - T \tan \theta \quad (i)$$

But  $\tan \theta$  is equal to the value of  $\frac{\partial y}{\partial x}$  at  $a$ , and  $\tan \theta'$  is equal to the value of  $\frac{\partial y}{\partial x}$  at  $b$ . Therefore  $\tan \theta' - \tan \theta$  is the increase of  $\frac{\partial y}{\partial x}$  from  $a$  to  $b$ , and this increase is  $\left(\frac{\partial^2 y}{\partial x^2}\right)dx$ . This is evident when we consider that  $\frac{\partial^2 y}{\partial x^2}$  means the rate of increase of  $\frac{\partial y}{\partial x}$  with respect to  $x$ . Therefore, substituting  $\left(\frac{\partial^2 y}{\partial x^2}\right)dx$  for  $\tan \theta' - \tan \theta$  in equation (i), we get

$$dF = T \frac{\partial^2 y}{\partial x^2} . dx \quad (ii)$$

But the net, unbalanced, upward force acting on  $ab$  must be equal to the mass of  $ab$  multiplied by the upward acceleration,  $\frac{\partial^2 y}{\partial t^2}$ , of  $ab$ . Therefore we have

$$m . dx \times \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} . dx$$

or

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\partial^2 y}{\partial x^2} \quad (28)$$

and this is identical to equation (27) if we put

$$v = \sqrt{\frac{T}{m}} \quad (29)$$

An equation identical in form to (28) is easily established as expressing the dynamics of the longitudinal motion of the air in a tube, of the water in a canal or of the material of a steel rod. See Franklin's *Electric Waves*, pages 154-158; published by Franklin and Charles, Bethlehem, Pa.

An equation identical in form to (28) is easily established as expressing the "dynamics" of electromagnetic motion on a transmission line. See Franklin and MacNutt's *Advanced Electricity and Magnetism*, pages 202-207; published by Franklin and Charles, Bethlehem, Pa.

**92. Pure wave on string traveling to the right.**—A particular solution of (28) is

$$y = f(x - vt) \quad (30)$$

where  $v$ , written for  $\sqrt{T/m}$ , is the velocity of the wave. This equation represents a bend of any shape whatever traveling to the right at velocity  $v$ .

Let  $V$  be the sidewise velocity of the wire at any point  $x$  as this bend travels by, and let  $s$  be the slope of the wire at the same point. Then  $V = \frac{\partial y}{\partial t} = -v \cdot f'(x - vt)$ , and  $s = \frac{\partial y}{\partial x} = +f'(x - vt)$ , so that

$$\frac{V}{s} = -v \quad (31)$$

*Any disturbance of the wire in which the sidewise velocity  $V$  of the wire is at each point equal to  $-vs$  where  $s$  is the slope of the wire at the same point is a pure wave traveling to the right, and a particular solution of (28).*

It is important to remember that  $v$  ( $= \sqrt{T/m}$ ) is the velocity of travel of a wave on the wire, and that  $V$  is the sidewise velocity of the wire which can have any moderate value whatever.

**93. Pure wave traveling to the left.** A particular solution of (28) is

$$y = F(y + vt) \quad (32)$$

and this represents a bend of any shape traveling to the left at velocity  $v$  ( $= \sqrt{T/m}$ ). In this case we have

$$\frac{V}{s} = +v \quad (33)$$

*Any disturbance of the wire in which the sidewise velocity  $V$  of the wire is at each point equal to  $+vs$  is a pure wave traveling to the left and a particular solution of (28).*

**Remark.**—Because of the opposite signs in equations (30) and (32) the algebraic sign of  $v$  is to be thought of as positive always; a wave traveling to the right is to be thought of as such, and a wave traveling to the left is to be thought of as such. That is to say, we are not to think of a wave traveling to the left as a wave traveling to the right at velocity  $-v$ . This is indeed desirable from the mathematical point of view, but from the physical point of view it is better to consider the actual facts.

**94. Satisfying of initial conditions.**—It is very easy, as shown in the following examples, to resolve any prescribed distribution of sidewise velocity and any prescribed distribution of slope each into two parts so that one part of the sidewise velocity together with one part of the slope may satisfy equation (31), and so that the other part of the sidewise velocity and the other part of the slope may satisfy equation (33) thereby establishing two particular solutions of (28) which together satisfy the prescribed initial conditions. The following example illustrates this, and in this example no boundary conditions exist.

An infinitely long stretched string is struck by a square faced hammer so as to impart a certain given sidewise velocity,  $2V$ , to the entire portion  $ab$  of the string as indicated in Fig. 161.

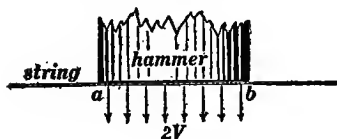


Fig. 161.

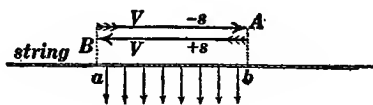


Fig. 162.

Note that  $2V$  is downwards and therefore to be considered as negative. Let the arrow  $A$ , Fig. 162 represent a combination of downward velocity  $V$  and slope  $+s$  satisfying equation (31), and let the arrow  $B$  represent a combination of downward velocity  $V$  and slope  $-s$  satisfying equation (33). Each of these arrows represents a solution of (28), and both together at the beginning represent the effect of the hammer blow. Therefore the two waves or wave pulses  $A$  and  $B$  give the actual motion of the string which is produced by the hammer blow. After lapse of  $t$  seconds  $A$  will have traveled a distance  $vt$  towards the right, and  $B$  will have traveled a distance  $vt$  towards the left; and the actual configuration of the string at the instant  $t$  (as to resultant sidewise velocity and slope) is easily determined by adding together the sidewise velocities and slopes which are associated with  $A$  and with  $B$  as indicated in Figs. 163 and 164.

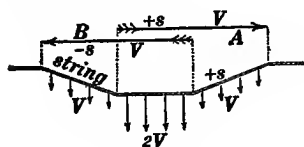


Fig. 163.

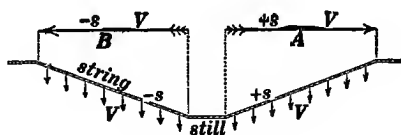


Fig. 164.

**95. Satisfying of boundary conditions.\*** The freedom or immovability of the end of a string determine the character of the reflection which takes place when a wave reaches the end of the string, *and the character of the reflection can always be specified so as to satisfy the boundary conditions.* The discussion of this matter is extremely simple for an electric wave on a transmission line,† but it is not so simple for waves on a string on account of

\* This discussion is limited to the two simplest cases, namely, (a) The case where the end of the string is immovable, and (b) The case where the end of the string is entirely free. The first case corresponds exactly to the open-ended transmission line and the second case corresponds exactly to the short-circuit-ended transmission line.

† See Franklin and MacNutt; *Advanced Electricity and Magnetism*, pages 212–216; published by Franklin and Charles, Bethlehem, Pa.

the highly artificial conditions which must be imagined to exist in order realize what is called a free end, whereas the "free end" is as simple and natural as the "immovable end" for the transmission line, for a steel rod and for the air column in a tube.

The pure wave on a string always involves the two elements *sidewise motion* and *slope* which travel along together and mutually sustain each other. Sidewise motion and slope are *opposite in sign* in a pure wave traveling to the right but *alike in sign* in a pure wave traveling to the left according to equations (31) and (33). Therefore when a wave is reflected at either end of a string  $V$  or  $s$  must change sign (but not both  $V$  and  $s$ ) because the direction of travel of the wave is reversed by reflection.

**Reversal of  $V$  at immovable end of a string.**—Let the heavy arrow in Fig. 165 represent a pure wave\* traveling towards the

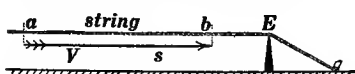


Fig. 165.

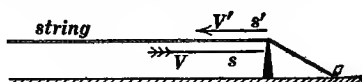


Fig. 166.

Reflection from immovable end of a string.

right towards the immovable end of the string at  $E$ . What happens when the wave  $ab$  reaches  $E$ ? Whatever happens *on the string* it must be equivalent to a pure wave traveling towards the right and a pure wave traveling to the left, one heaped on top of the other (superposed), because this is the physical interpretation of equation (i) of Art. 90 which is the general solution of equation (28), and whatever happens *at the extreme end of the string* it is only necessary that the actual sidewise velocity of the string be zero there.

Therefore if we assume a pure wave (sidewise velocity  $V'$  and slope  $s'$ ) to shoot out from  $E$  towards the left, and if  $V' + V$  is always equal to zero at  $E$ , we must have the correct

\* We here think of  $V$  and  $s$  as having the same values all along  $ab$ , but this restriction is not necessary for present purposes.

solution because all necessary conditions are satisfied. That is to say, we must have

$$\frac{V}{s} = -v \quad (i)$$

and

$$\frac{V'}{s'} = +v \quad (ii)$$

according to equations (31) and (33); and we must have

$$V + V' = 0 \quad (iii)$$

From these three equations we get  $s' = s$  and  $V' = -V$ ; which means (a) That the reflection is complete because numerically  $s' = s$  and  $V' = V$ , and (b) That the sidewise velocity in the reflected wave is opposite to sidewise velocity in the original wave. This is called the *reversal of  $V$  by reflection*.

**Reversal of  $s$  by reflection at free end of string.**—Figure 167

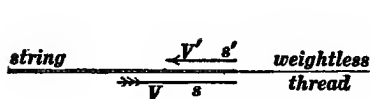


Fig. 167.

Reflection from free end of string.

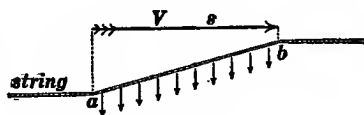


Fig. 168.

represents an ideal arrangement in which the end of an actual string is held by an ideal weightless (mass-less) thread. In this case the end of the string is free to move sidewise, but there can be no slope at the extreme end of the string. Therefore we must have

$$\frac{V}{s} = -\frac{V'}{s'}$$

and

$$s + s' = 0$$

from which we get  $V' = V$  and  $s' = -s$ . Which means (a) That the reflection is complete, and (b) That the slope in the reflected wave is opposite in sign to the slope in the original wave. This is called the *reversal of  $s$  by reflection*.

Figure 168 shows the actual configuration of the string in a wave traveling to the right for the case in which  $V$  and  $s$  have same values throughout the wave and where  $V$  is downwards.

**96. Motion of a plucked string.**—As an example of a problem involving both initial and boundary conditions let us consider the motion of a plucked string. A string is pulled to one side at its middle point, as indicated by  $APB$  in Fig. 169, and re-

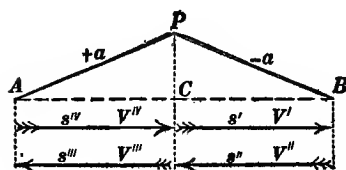


Fig. 169

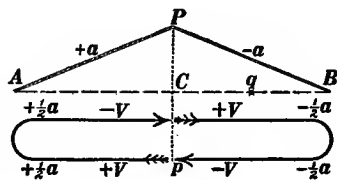


Fig. 170.

leased; and it is required to determine the motion of the string. The initial condition of the string may be resolved into the four pure waves  $V's'$ ,  $V''s''$ ,  $V'''s'''$  and  $V^{iv}s^{iv}$  such that

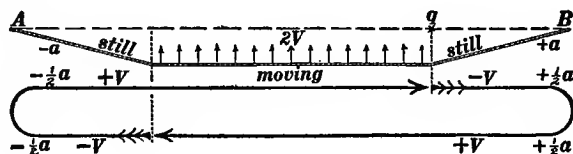


Fig. 171.

$$s' + s'' = -a \quad (i)$$

$$V' + V'' = 0 \quad (ii)$$

$$s''' + s^{iv} = +a \quad (iii)$$

$$V''' + V^{iv} = 0 \quad (iv)$$

and, according to equations (31) and (33), we must have

$$\frac{V'}{s'} = \frac{V^{iv}}{s^{iv}} = -v \quad (v) \text{ and } (vi)$$

and

$$\frac{V''}{s''} = \frac{V'''}{s'''} = +v \quad (vii) \text{ and } (viii)$$

where  $v(=\sqrt{T/m})$  is the velocity of travel of the waves on the string. Therefore  $s', s'', s''', s^{iv}, V', V'', V'''$  and  $V^{iv}$  are determined.

In Fig. 170  $V$  represents the common numerical value of  $V', V'', V'''$  and  $V^{iv}$ , and  $s$  represents the common numerical value of  $s', s'', s'''$  and  $s^{iv}$ . Also in Fig. 170 the two waves  $V's'$  and  $V''s''$  are represented as a single wave because  $V's'$  merges into  $V''s''$  on reflection (reversal of  $V$ ), and the two waves  $V'''s'''$  and  $V^{iv}s^{iv}$  are represented as a single wave for the same reason. *The boundary conditions are satisfied by always specifying the proper kind of reflection as the two waves in Fig. 170 travel back and forth along the string.*

Figure 171 shows the state of things after each wave in Fig. 170 has traveled over a distance  $1\frac{1}{4}l$  where  $l$  is the length of the string.

**97. Behavior of a rod which strikes endwise against a rigid wall.**—A longitudinal wave on a steel rod consists of a state of compression  $C$  (meaning stress in dynes per square centimeter or “pounds” per square foot; negative compression being a tension) and a state of actual forward or backward motion of the material of the rod at velocity  $V$ —both traveling along together at very great velocity  $v$  and mutually sustaining each other. The mathematics of this kind of wave motion is identical in every detail to the mathematics of transverse wave motion on a string, and it is sufficient therefore to point out the following:

(a) The wave velocity  $v$  is

$$v = \sqrt{E/D} \quad (34)$$

where  $E$  is the stretch modulus of the steel in “pounds” per square foot), and  $D$  is the density of the steel in slugs per cubic foot.

(b) In a pure wave traveling to the right we have

$$\frac{V}{C} = +a \quad (35)$$

where  $a$  is a constant

(c) In a pure wave traveling to the left we have

$$\frac{V}{C} = -a \quad (36)$$

**Remarks.**—The positive sign in (35) becomes negative if we choose to consider *tension* rather than *compression* as *positive*.

The quantity  $a$  is not the wave velocity because we have chosen the *stress*  $C$  rather than the corresponding *strain* which would be completely analogous to slope  $s$  in the case of a stretched wire. The quantity  $a$  is equal to wave velocity divided by stretch modulus  $E$ . That is

$$a = \frac{v}{E} \quad (37)$$

When a steel rod moving towards the right at velocity  $V$  strikes a rigid wall, the subsequent motion of the rod must be expressible as two particular solutions of equation (27) superposed. Now the original uniform velocity of the rod is such a solution as may be easily shown. Therefore let us symbolize this solution by the heavy dotted line in Figs. 172 and 173.

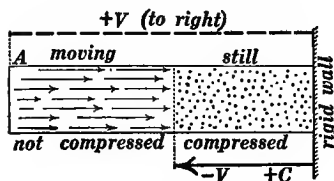


Fig. 172.

Steel rod striking a wall.

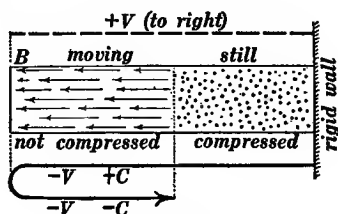


Fig. 173.

Steel rod rebounding from wall.

It remains to devise another particular solution of (27) which when added to (superposed upon) the uniform  $V$  will satisfy the boundary conditions (at the wall). The required particular solution is an indefinitely-long-drawn-out wave shooting out along the rod from the wall. This wave we may call a *ribbon wave* because it suggests a ribbon coming continuously out of the mouth of a prestidigitateur, and because it has the same intensity

all along (same values of  $V$  and same values of  $C$ ). This ribbon wave is represented by the heavy arrow in Figs. 172 and 173. The velocity of the material of the rod in the ribbon wave must be equal and opposite to the original velocity  $V$  of the moving rod *because the velocity of the material at the wall-end of the rod must be zero*. Therefore  $-V$  is the velocity of the material of the rod which is associated with the ribbon wave, and, from equation (36) we find that  $C$  must be positive (a compressive stress) and the value of  $C$  is  $V/a$  or  $VE/v$  according to equations (36) and (37).

The ribbon wave in Fig. 172 is reflected from the free end of the rod with reversal of  $C$  as indicated in Fig. 173.

The first lap of the ribbon wave wipes out the velocity of the rod and lays down a compressive stress  $C$ . The second lap of the ribbon wave wipes out this compressive stress and lays down velocity  $-V$ , and when the second lap of the ribbon wave reaches the wall the steel rod is moving at uniform velocity  $V$  away from the wall and without any stress in it anywhere. If the steel rod were imagined to become stuck fast to the wall, the second lap of the ribbon wave would be reflected from the wall with reversal of  $V$ , then the third lap would be reflected from the free end with reversal of  $C$ , and so on indefinitely if there were no energy losses; the ribbon wave meantime continuing to shoot out from the wall-end. It is not worth while to describe the details because they are essentially identical to the details of behavior of the air in a tube as described in the next article.

**98. A tube which contains slightly compressed air is suddenly opened at one end and it is required to determine the motion of the air in the tube.**—The initial condition of uniform compression is itself a solution of equation (27) and it is symbolized by the heavy dotted line in Figs. 174 and 175. When the end of the tube is opened the pressure must fall instantly to normal atmospheric pressure and always keep that value.\*

\* The motion of the outside air produces a complication which is here neglected. This matter has been considered in great detail by Helmholtz—the Helmholtz end-effect.

A ribbon wave shoots inwards from the opened end of the tube, and the rarefaction (negative compression —  $C$ ) in this ribbon wave brings the pressure of the air at the opened end of the tube to normal atmospheric pressure.

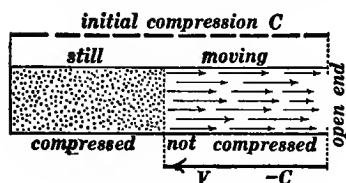


Fig. 174.

First stage of motion of air in tube when end is opened.

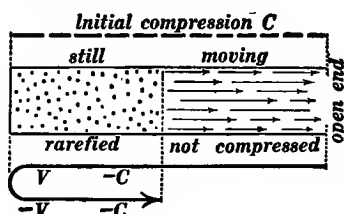


Fig. 175.

Second stage of motion of air in tube when end is opened.

The first lap of the ribbon wave (first stage of motion of the air in the tube) wipes out the initial compression and lays down a condition of uniform velocity towards the right as indicated in Fig. 174.

The ribbon wave is reflected from the closed end of the tube with reversal of  $V$ , and the second lap of the ribbon wave (second stage of motion of the air in the tube) wipes out the uniform velocity previously laid down and lays down a condition of uniform rarefaction as indicated in Fig. 175.

The ribbon wave is then reflected from the open end of the tube with reversal of  $C$ , and the third lap produces a third stage of motion of the air in the tube, and so on indefinitely if there are no energy losses. Four laps of the ribbon wave brings the air in the tube to its initial uniformly compressed condition and one complete vibration of the air is said to have taken place.

#### PROBLEMS.

**152.** A string one meter long having a tension of 20,000,000 dynes and a mass of 0.04 gram per centimeter is plucked at a point 25 centimeters from one end, this point of the string being forced sidewise 1.5 centimeters and released. Find the exact

shape of the string after 0.001 second and specify what parts of the string are moving, in what direction they are moving and how fast they are moving.

**153.** Two steel rails each 30 feet long moving in opposite directions at velocities of 15 feet per second strike together end-wise. Find exact condition of each rail as to motion and stress 0.001 second after the collision. Calculate wave velocity, and other necessary data from stretch modulus and density of the steel.

**154.** Trace a wave back and forth on a steel rod (free at both ends) and show that the wave comes back to its initial position and exactly to its initial condition (no energy losses) after traveling over twice the length of the rod. Calculate the wave velocity along the steel rod from values of stretch modulus and density, and find time required for a steel rod 6 feet long to make one complete longitudinal vibration, rod being free at both ends.

## APPENDIX A.

### SIMPLE MEASUREMENTS AND EXPERIMENTS.

**1. The laboratory point of view.**—It is of the utmost importance that the study of a purely theoretical text, such as this text-book on mechanics, should not blunt the students' appreciation of the fact that all calculations in the physical sciences are based on observed data.

"Our method," says Bacon, "is to dwell among things soberly, without abstracting or setting the mind farther from them than makes their images meet; and the capital precept for the whole undertaking is that the eye of the mind be never taken off from things themselves, but receive their images as they truly are, and God forbid that we should offer the dreams of fancy for a model of the world."

The purpose of this appendix is to emphasize the laboratory point of view: (*a*) By giving a few detached statements relating to fundamental measurements in mechanics, and (*b*) By pointing out portions of the foregoing text which have to do with particular measurements in the laboratory.

**2. The measurement of time.**—The beginner in the physical laboratory is usually very careless in making time measurements and consequently a few suggestions as to methods and a little supervision of his preliminary practice are very important.

In manipulating a revolution counter or in actually counting the vibrations of a pendulum, the observer cannot look at a watch or clock. He must use a stop-watch, or have the assistance of a helper. In the latter case a clock should be used which beats seconds or half-seconds so that the helper can give his time signals accurately. An ordinary watch which beats fifth-seconds is very unsatisfactory.

The helper should look at the second hand, catch the seconds or half-seconds count, say *ready* about two seconds before the time signal, and say *now* when the time signal is to be given; *and these words should be spoken very sharply*. Snappy time signals are very necessary. With care a time interval can be set off more accurately in this way than by means of a stop-watch.

There is always a very considerable error, a lag error, in the giving of a time signal by a helper and in the responding or reacting to the signal by an observer; but this lag error affects the beginning and ending of a time interval to about the same extent *unless the observer has to do very different things when the respective signals are given*. An error of two or three tenths of a second is unavoidable in the setting off of a time interval by the above method, and therefore a five-minute interval is the shortest interval that can be set off with a precision of one tenth of one percent.

The error in rate of a watch or clock is usually entirely negligible in comparison with what we may call "setting off" errors. Thus a watch which gains or loses one minute a day is a poorly adjusted watch, and yet one minute is only  $1/1400$  of a day, or less than one-tenth of one per cent.

**3. The use of the reading telescope for projecting a vertical distance (to be measured) upon a near-by scale.**—The High School graduate can be brought easily to a fair degree of precision in ordinary length measurements and in the use of the balance, but the important use of the reading telescope for measuring vertical distances\* demands special attention.

Suppose, for example, that one wishes to measure, with a fair degree of precision, the difference in level of water or mercury in two adjacent tubes *a* and *b*. The procedure is as follows:

A telescope with cross-hairs is mounted on a slider which can be moved up or down on a vertical pillar or column so that the telescope can be brought to any desired height, and when the slider is clamped the telescope is free to turn about the vertical pillar as an axis.

\* The use of the cathetometer is, of course, out of the question.

Place the scale  $s$  in a vertical position near tubes  $a$  and  $b$ , and arrange so that the supporting pillar of the reading telescope is equidistant from  $a$ ,  $b$  and  $s$ .

With the telescope nearly level move the slider up or down until the telescope can be sighted at the water level in  $a$ , then clamp the slider, raise or lower the slider by a slow-motion screw (or slightly tilt the telescope by a slow-motion screw) until the cross-hair of the telescope is accurately coincident with the water level in  $a$ ; *then turn the telescope carefully until the scale comes into the field of view, and read the position of the cross-hair on the scale.*

Repeat with respect to water level in tube  $b$ .

The difference of the two scale readings is the required difference in level.

The telescope must be very nearly equidistant from  $a$ ,  $b$  and  $s$  so as to remain sharply in focus when turned from one to the other—adjustment of focus when turning the telescope from one to the other is decidedly not permissible. Why?

What error is introduced by inclination of pillar towards or away from  $a$ ,  $b$  and  $s$ ? What error when pillar is inclined sidewise? What error when telescope is not exactly level,  $a$ ,  $b$  and  $s$  not being exactly equidistant from the telescope?

**4. Determination of friction torque in a rotating ball bearing carrying a given load.\***—Nothing with which the engineer deals, perhaps, is so uncertain in value, so unexpectedly variable, as friction, and therefore a very low degree of precision, only, is necessary in measuring friction. The following experiment, crude as it is, is eminently practical.

A horizontal spindle is carried by a ball bearing, the bearing itself being attached to a very rigid support, a heavy weight (to give the desired load) is fixed to the spindle, pendulum-fashion, by a short rod, and the weight is set oscillating.

It is easy to make measurements which will enable one to

\* The method here described was first used, so far as known, by W. S. Franklin in some ball-bearing tests at the Bureau of Standards in July, 1918.

calculate the amount of energy (potential energy) given to the pendulum at the start. It is easy to estimate the angular amplitude  $a$  of the pendulum at the start, and again,  $a'$ , after it has completed  $n$  vibrations. The average angular amplitude is then  $\frac{1}{2}(a + a')$ , and the total angular travel is  $\frac{1}{2}(a + a')n$ .

It is easy to estimate the energy left in the pendulum at the end of the  $n$  vibrations (most easily calculated as potential energy). Therefore loss of energy is known and this is equal to  $\frac{1}{2}(a + a')nT$ , where  $T$  is the friction torque.

Friction of the air is here neglected; and it is, in fact, negligible, considering the desired precision.

**5. Measurement of the velocity of a bullet by means of the ballistic pendulum.**—A convenient form of ballistic pendulum is shown in Fig. 75 of chapter II. The bullet is shot into the wooden block  $B$  which is backed by a steel plate, and the velocity of the bullet may be calculated with the help of equation (iii) of Art. 38.

Several bullets must be extracted from sample cartridges and weighed, and the mass so determined must be used for  $m$  in equation (iii); the mass  $M$  of the pendulum is easily determined by weighing.

The velocity  $v$  of the pendulum immediately after the impact of the bullet is most conveniently calculated from the measured length  $l$  in Fig. 75 together with the observed distance over which a very light slider is pushed along a horizontal bar by the pendulum, and the known acceleration of gravity.

A 22-caliber rifle, preferably with a Maxim silencer, is suitable for this experiment; and an interesting detail is to note the decrease of velocity of bullet with successive shots when the rifle is not cleaned after each shot.

**6. Study of harmonic motion.**—A ball of mass  $m$  is hung by a helical spring, and the elongation  $l$  of the spring due to the stretching force  $mg$  is observed; then the "stiffness coefficient" of the spring (see Art. 43) is  $k = mg/l$ .

The ball is then set oscillating up and down, and the number

of complete oscillations per second,  $n$ , is determined. Then  $k = 4\pi^2 n^2 m$ , according to equation (6) of Art. 46.

Then we have  $\frac{mg}{l} = 4\pi^2 n^2 m$  from which the acceleration of gravity may be calculated.

**Remark.**—The determination of the acceleration of gravity by observing the frequency of the vibrations of a simple pendulum is important, but it need not be discussed here.

**7. Experiment with a rolling disk.**—Given a straight smooth inclined track (two rails) on which a disk and axle can roll. Having a clock which beats seconds audibly, the disk can be released on a beat of the clock by quickly jerking a heavy block from in front of the disk,\* and after several trials a second block can be placed on the track to mark the point reached by the rolling disk after a whole number of seconds.

If the disk is properly made its radius of gyration  $\rho$  can be calculated from its measured dimensions and then the acceleration of gravity can be calculated from the known value of  $\rho$ , the observed distance rolled by the disk in a known time, and the measured slope of the track.

**8. Determination of coefficient of torsion of a wire.**—A disk of which the moment of inertia has been calculated from its measured dimensions and mass is hung by a steel wire, the axis of figure of the disk being coincident with the wire. The frequency,  $n$ , of the rotatory oscillations of the disk can then be found, and the "stiffness coefficient" or *coefficient of torsion* of the wire can be calculated with the help of equation (13) of Art. 57.

**Remark.**—The coefficient of torsion of a wire is the coefficient of  $\alpha$  in equation (26) of Art. 85; and therefore, if the value of this coefficient has been determined as above explained, we can calculate the slide modulus  $n$ , equation (26), if we know the radius  $r$  and the length  $L$  of the steel suspending wire.

\* A series of feints should be made with the hand as if to jerk the block away, these feints being in rhythm with the beats of the clock, and when the rhythm seems to be complete the operation of jerking can be carried out.

**9. Experimental determination of the moment of inertia of a body.**—(a) If a wheel and axle is allowed to roll down an inclined track as explained above in connection with the experiment with a rolling disk, the radius of gyration of the wheel and axle can be calculated if the acceleration of gravity is known, friction being, of course, ignored.

(b) Having determined the coefficient of torsion of a suspending wire as explained above, a body of unknown moment of inertia can be hung by the wire, the frequency of its vibrations determined, and its moment of inertia calculated with the help of equation (13) of Art. 57.

(c) A body, such as a connecting rod, for example, may be swing as a gravity pendulum as indicated in Fig. 98 of Art. 58, the distance  $l$  from its center of gravity to the axis of support can be measured, and the frequency of its vibrations determined. The radius of gyration  $K'/m$  of the body with reference to the axis through  $O$  can then be calculated, the acceleration of gravity being known. Then  $K'$  can be calculated,  $m$  being known, and then the moment of inertia of the body about an axis through  $C$  parallel to the axis through  $O$  can be calculated with the help of equation (11) of Art. 53.

**10. The open tube manometer.**—One of the commonest requirements of engineering practice is to determine a difference of hydrostatic pressure by measuring the height of a column of water or mercury which balances it. In practice it is usually necessary to do the best one can to measure this height by means of a simple scale; but in the laboratory it is often desirable to use the reading telescope as explained in Art. 3 of this appendix.

The value of the measured pressure difference in c.g.s. units or in f.s.s. units is given by equation (16) of Art. 62.

**Remark.**—The use of the Venturi meter and the use of the Pitot meter are explained in Art. 76 of Chapter V.

**11. Measurement of specific gravity.**—The simpler methods for determining the volume of a vessel and for finding the specific

gravity of a substance are explained in Art. 66 of Chapter IV, and the method of reducing from specific gravity to density is explained under example (b) of Art. 66.

**12. Determination of the stretch modulus of a substance.—**

(a) The simplest method is to observe the elongation  $\Delta l$  of a wire whose initial length is  $l$  and whose sectional area is  $q$ ,  $F$  being the stretching force; and calculate  $E$  with the help of equation (22) of Art. 79.

(b) An easier method, experimentally, is to arrange a beam as indicated in Fig. 151,\* measure breadth  $b$  and depth  $d$  of the beam, measure the force  $F$  and the distances  $L$  and  $L'$ , and measure the upward movement  $h$  of the middle of the beam; and then calculate  $E$  with the help of equation (24) of Art. 84.

**Remark.**—The determination of the slide modulus of a steel wire is explained in Art. 8 of this appendix.

\* The equation of deflection of a beam loaded at the middle is not derived in this text.

## APPENDIX B.

### ERRORS OF MEASUREMENT.

Physical measurements are always subject to error, and in accurate work means must be devised: (*a*) for reducing error as much as possible, and (*b*) for estimating the probable magnitude of the residual or uncorrected error in order to be able to estimate the degree of precision of a result. Elimination of error and estimation of probable magnitude of outstanding error, in so far as these two things can be discussed in fairly general terms, are the objects of the following discussion.

**13. Systematic errors.**—In weighing a body repeatedly the same errors will be introduced every time by buoyancy of air by inequality of arms of balance and by the errors of the weights themselves. If a distance is measured by a meter scale which is warmer than it ought to be and therefore too long, there will be an error in the measured distance, and this error will be always the same under the given conditions. If the scale of an ammeter is faulty in its divisions, an electric current as measured by the ammeter will be in error, and the error will always be the same under the given conditions. Such errors are called *regular errors* or *systematic errors*.

**14. Erratic errors.**—When a measurement is repeated with great care there are always irregular discrepancies between successive measurements. Suppose, for example, that two meter bars are laid end-to-end stepwise along a smooth hard floor so as to measure the distance between two fine lines 10 or 15 meters apart. If one were content to get the distance to the nearest decimeter one could easily get the same result every time the measurement is repeated, but if one tries to get a result to 0.01 centimeter, using a magnifying glass to read the scales and using utmost care throughout, one would be likely to get a

slightly different result every time the measurement is repeated, and these differences would be irregular or erratic (never twice alike). Such errors are called *irregular errors* or *erratic errors*.

**15. Elimination of systematic error.**—Physical laboratory manuals are largely devoted to methods for eliminating systematic errors in different kinds of physical measurements, and it is evident that only an extremely inadequate statement of this matter can be given here.

(a) *Instrumental errors.* No instrument is ever made or adjusted with such precision as to be equal to the requirements of very accurate measurement, it is always better to study or test the instrument and make allowances for incomplete adjustments. Thus the two arms of an ordinary chemical balance are never nearly enough of the same length for extremely accurate weighing, the most carefully kept ship's chronometer always runs a little fast or slow, a very high-grade thermometer is sure to have an uneven bore, etc.

Insofar as instrumental errors are due to the incomplete adjustment or to an imperfect scale (as on a pressure gauge or ammeter, or thermometer), these errors must be eliminated either:

(1) By a properly planned procedure, or

(2) By a careful study of the imperfectly adjusted instrument and the determination of data from which the error can be calculated, or

(3) By careful calibration.

The first is illustrated by the procedure called "double weighing" for eliminating the error due to the inequality of the arms of a balance.

The second may be illustrated as follows: A chronometer is found to gain 8.75 seconds per day by comparing it with the daily time signals from Washington, and therefore the true value of any time interval is  $\frac{86400}{86400 + 8.75} \times (R_1 - R_2)$  seconds, where  $R_1$  and  $R_2$  are two readings of the chronometer.

The third is illustrated by a pressure gauge, or by an ammeter, or by an ordinary thermometer. Such instruments must be carefully calibrated for precise work.

(b) *Condition errors.* Usually an instrument should be used under standard conditions, but in many cases it is impracticable to bring about the standard conditions. A good example is the use of a surveyor's tape for measuring a base line. The surveyor knows that his tape is correct at a certain temperature; but he cannot control the weather, he must note the temperature of the tape when the measurement is made, and make allowance for the expansion of the tape. A mercurial barometer should be at a standard temperature, and if it is not, allowances must be made for the expansion of the mercury and for the expansion of the scale. Accurate weighing should be carried out in a vacuum, but this is impracticable, and therefore allowance must be made for the buoyant force of the air on the weights and for the buoyant force of the air on the weighed body.

In many cases the standard conditions under which an instrument should be used are easily realized. Thus a mercurial barometer should be vertical, an ammeter or voltmeter should be horizontal, no magnets or unnecessary electric wires should be near an ammeter or voltmeter, etc.

*Estimation of outstanding instrumental and condition errors.* It is possible in most cases to estimate the uncertainty of instrumental and condition allowances, but this estimation of outstanding instrumental and condition error is in every case a highly specialized problem, and it cannot be discussed in general terms; any further discussion of this matter would therefore be out of place here.

**16. How to make erratic error small, and how to estimate the probable value of erratic error.** The only way to make erratic error small is to take great care in all manipulations, to keep conditions as steady as possible while a measurement is being made, to make very accurate scale readings, and to repeat a measurement a number of times and take the mean.

The only way to estimate the probable magnitude of erratic error is by considering the differences between repeated measurements.

Certain important ideas relating to erratic error are most easily explained by considering the shooting at a target. Various effects give rise to systematic errors in the placing of the shots on a target. Thus gravity pulls all of the shots downwards and a side wind carries all of the shots to one side. We may, however, take the "center of gravity" of all the shots as the center of the target, and the scattering of the shots with respect to this center represents the erratic errors of the shots.

**Probable error of a single shot.** Let a circle be drawn about the center of the target so as to include one-half of a large number,  $n$ , of shots, the other half being outside of the circle, and let the radius of this circle be  $r$ ; then, if another shot be made, it is an even chance that this shot will strike inside or outside of the circle, and therefore the radius  $r$  of the circle is called *the probable error of a single shot*. The theory of probability shows that

$$r = 0.6745 \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + d_4^2 + \dots}{n - 1}}$$

where  $n$  is the number of shots and the  $d$ 's are the distances of the individual shots from the center of the target.

**Probable error of a single observation.** A certain measurement is repeated  $n$  times. Then if another measurement is taken it is an even chance that this additional measurement will differ from the mean of the  $n$  measurements by *more or less* than  $r$ , where

$$\left\{ \begin{array}{l} \text{The probable error} \\ \text{of a single} \\ \text{measurement} \end{array} \right\} = r = \pm 0.6745 \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + d_4^2 + \text{etc.}}{n - 1}} \quad (1)$$

where the  $d$ 's are the differences between the individual measurements and their mean; these  $d$ 's are called the *residuals*, and the quantity  $r$  is called the *probable error of a single observation*.

**Probable error of the mean of a number of observations.**

If the  $n$  measurements be repeated, it is an even chance that the mean of the  $n$  repeated measurements will differ from the mean of the original  $n$  measurements by *more or less* than  $r/\sqrt{n}$ , and this quantity is therefore called the *probable error of the mean of the  $n$  measurements*. That is

$$\left\{ \begin{array}{c} \text{The probable error} \\ \text{of the mean of} \\ n \text{ measurements} \end{array} \right\} = \pm \frac{r}{\sqrt{n}} \quad (2)$$

**Example.** A base line was measured ten times with the following results:

Individual measurements	Residuals
124.327 meters	+ 0.0022 = $d_1$
124.324    "	- 0.0008 = $d_2$
124.331    "	+ 0.0062 = $d_3$
124.332    "	- 0.0028 = $d_4$
124.332    "	+ 0.0072 = $d_5$
124.323    "	- 0.0018 = $d_6$
124.320    "	- 0.0048 = $d_7$
124.318    "	- 0.0068 = $d_8$
124.326    "	+ 0.0012 = $d_9$
124.325    "	+ 0.0002 = $d_{10}$
<hr/>	
mean 124.3248	

The probable error of each single measurement, as calculated by the above formula, is  $\pm 0.0044$  meter, and the probable error of the mean of the ten measurements is  $\pm 0.0014$  meter.

Now the average of the residuals, regardless of sign, is  $\pm 0.0034$  meter, and this is nearly as good an estimate of the probable error of a single measurement as  $\pm 0.0044$  meter. Therefore *the average of the residuals (regardless of sign) is nearly always used instead of the correctly calculated probable error of a single observation, and the average of the residuals divided by  $\sqrt{n}$  is nearly always used instead of the correctly calculated probable error of the mean of a set of measured values; and these are usually spoken of as probable errors because this term suggests so clear a meaning.*

**17. Influence of errors in observed values on a calculated result.** Let  $R$  be a result which is calculated from, let us say, three measured quantities  $x$ ,  $y$  and  $z$ . An equation expressing  $R$  in terms of  $x$ ,  $y$  and  $z$  is, of course, given. Differentiating this equation, we get expressions for  $\frac{\partial R}{\partial x}$ ,  $\frac{\partial R}{\partial y}$  and  $\frac{\partial R}{\partial z}$ ; these expressions involve  $x$ ,  $y$  and  $z$ , and, using the measured values of  $x$ ,  $y$  and  $z$ , the numerical values of  $\frac{\partial R}{\partial x}$ ,  $\frac{\partial R}{\partial y}$  and  $\frac{\partial R}{\partial z}$  are known. Then, representing the probable errors of  $x$ ,  $y$  and  $z$  by  $dx$ ,  $dy$  and  $dz$ , we have

$$\begin{aligned} dR_x &= \frac{\partial R}{\partial x} \cdot dx \\ dR_y &= \frac{\partial R}{\partial y} \cdot dy \\ dR_z &= \frac{\partial R}{\partial z} \cdot dz \end{aligned} \tag{3}$$

where  $dR_x$  is the probable error in  $R$  due to the error in  $x$ ,  $dR_y$  is the probable error in  $R$  due to the probable error in  $y$ , and  $dR_z$  is the probable error in  $R$  due to the probable error in  $z$ .

The errors  $dR_x$ ,  $dR_y$  and  $dR_z$  are called the *partial errors* or *component errors* in  $R$ , and the question now is 'What is the *total error* or *resultant error* in  $R$  likely to be?' If the component errors should happen to be all of the same sign, the resultant error or total error would be equal to their sum; *but the most probable value of the resultant error is*

$$dR = \sqrt{(dR_x)^2 + (dR_y)^2 + (dR_z)^2} \tag{4}$$

*Example.* A body is weighed in air, giving  $x = 243.60$  grams with a probable error  $dx = \pm 0.016$  gram; and the body is weighed in water, giving  $y = 218.44$  grams with a probable error  $dy = \pm 0.025$  gram. The specific gravity of the body is given by the equation

$$R = \frac{x}{x - y} = 9.682$$

Using the measured values of  $x$  and  $y$ , we get  $\frac{\partial R}{\partial x} = 0.34$  and  $\frac{\partial R}{\partial y} = 0.39$ . Therefore  $dR_x = \pm 0.0054$  and  $dR_y = \pm 0.0098$ ; and the resultant probable error of the result is  $dR = \pm 0.011$ . That is to say, the result is  $9.682 \pm 0.011$ .

**18. When is one component error negligible in comparison with another?** Consider a result  $R$  which is calculated from two measured quantities  $x$  and  $y$ . Then, using the above notation, we have

$$dR = \sqrt{(dR_x)^2 + (dR_y)^2}$$

and if the component error  $dR_y$  is negligible in comparison with  $dR_x$ , the resultant error  $dR$  will be sensibly equal to  $dR_x$ , or, equal, let us say, to 1.1 times  $dR_x$ . Therefore, using this value for  $dR$  in the above equation, we get

$$1.1 \times dR_x = \sqrt{(dR_x)^2 + (dR_y)^2}$$

from which we get

$$dR_y = 0.46 \times dR_x$$

*Therefore, one component error in a result is negligible in comparison with another if it is a little less than half as large.*

In the same way it can be shown that two equal component errors of  $dR_y$  and  $dR_z$  are negligible in comparison with  $dR_x$  if they are each about three-tenths of  $dR_x$ .

**19. Errors in measured quantities which lead to negligible errors in a result.** Consider a result  $R$  which is calculated from several measured quantities  $x$ ,  $y$  and  $z$ . It often happens that some of the measured quantities need to be measured with great precision whereas the others do not need to be measured with great precision. This matter can be best explained by giving actual examples.

*Example I.* The length  $L$ , the diameter  $D$  and the mass  $m$  of a slender brass <sup>rod</sup> are measured, and the density  $R$  of the rod is calculated by using the equation

$$R = \frac{4m}{\pi D^2 L}$$

The approximate measured values are  $L = 101$  centimeters,  $D = 0.125$  centimeter and  $m = 105$  grams. The most careful measurement of  $D$  is subject to a probable error  $dD = \pm 0.001$ , and it is required to find how large the probable errors  $dL$  and  $dm$  may be without appreciably affecting the resultant error  $dR$ . Using the measured values of  $L$ ,  $D$  and  $m$ , we find

$$\frac{\partial R}{\partial L} = 0.84, \quad \frac{\partial R}{\partial D} = 1420 \quad \text{and} \quad \frac{\partial R}{\partial m} = 0.81$$

Therefore the component error  $dR_D = 1420 \times 0.001 = 1.42$ , and, to be negligible the two errors  $dR_m$  and  $dR_L$  should each be about three tenths of  $dR_D$  according to Art. 18, that is

$dR_m = 0.42 = 0.81 \times dm$ , whence  $dm = \pm 0.5$  gram  
and

$$dR_L = 0.42 = 0.84 \times dL, \text{ whence } dL = \pm 0.05 \text{ cm.}$$

Therefore if an error of  $\pm 0.001$  cm. is unavoidable in the measurement of the diameter of the rod, it is useless to measure  $m$  more accurately than to the nearest gram, and it is useless to measure  $L$  more accurately than to the nearest millimeter!

This sort of thing is encountered in nearly every indirect measurement (where a result is calculated from several measured quantities); some of the measured quantities must be determined with the greatest possible precision, and any effort which might be expended in determining the other measured quantities beyond a certain easily attained accuracy would be wasted effort.

*Example II.* The specific heat of copper is determined by dropping  $M$  grams of copper at  $h^\circ$  C. into  $W$  grams of cool water at temperature  $c^\circ$  C., and observing the resultant temperature  $r^\circ$  C. The result (specific heat of the copper) is

$$R = \frac{W(r - c)}{M(h - r)} \quad (\text{ii})$$

this formula is taken in its simplest form in order that the following discussion may not be unnecessarily complicated.

The observed values and probable errors are as follows:

$$\begin{aligned} W &= 200 \text{ grams } \pm dW \\ M &= 30 \text{ grams } \pm dM \\ h &= 99^\circ \text{ C. } \pm dh \\ c &= 16.85^\circ \text{ C. } \pm 0.01^\circ \\ r &= 18.00^\circ \text{ C. } \pm 0.01^\circ \end{aligned}$$

The two temperatures  $c$  and  $h$  are measured by the same thermometer and both are subject to the same error which we will take as  $\pm 0.01$  centigrade degree.

Let us consider how large the errors  $dW$ ,  $dM$  and  $dh$  may be and leave, as the only appreciable error in the result, the error due to  $dc$  and  $dr$ .

Differentiating equation (ii) and using the above values for  $W$ ,  $M$ ,  $h$ ,  $c$  and  $r$ , we get, disregarding algebraic signs,

$$\frac{\partial R}{\partial W} = 0.00047, \quad \frac{\partial R}{\partial M} = 0.0032, \quad \frac{\partial R}{\partial h} = 0.0012$$

$$\frac{\partial R}{\partial c} = 0.083 \quad \text{and} \quad \frac{\partial R}{\partial r} = 0.084.$$

Therefore the probable error  $dR_c = \pm 0.00083$ , the probable error  $dR_r = 0.00084$ , and the resultant error due to both is  $\pm 0.0012$ .

We may consider the errors  $dR_w$ ,  $dR_M$  and  $dR_h$  as negligible if each is about two tenths of 0.0012, or if each is equal to  $\pm 0.00024$ . Therefore we have

$$0.00024 = \frac{\partial R}{\partial W} \times dW; \quad \text{whence } dW = \pm 0.51 \text{ gram}$$

$$0.00024 = \frac{\partial R}{\partial m} \times dM; \quad \text{whence } dM = \pm 0.075 \text{ gram}$$

and

$$0.00024 = \frac{\partial R}{\partial h} \pm dh; \quad \text{whence } dh = \pm 0.20 \text{ degree.}$$

Therefore, for the given values of  $W$ ,  $M$ ,  $h$  and  $c$ , it would be wasted effort to measure  $W$  nearer than  $\pm 0.51$  gram, or to measure  $M$  nearer than  $\pm 0.075$  gram, or to measure  $h$  nearer than  $\pm 0.2^\circ$ , if  $c$  and  $r$  can be measured only to  $0.01$  degree.

**20. Percentage error.** The actual value of the probable error is a wholly satisfactory measure or estimate of the precision of a result, but when the precision of one result is to be compared with the precision of a different result it is desirable to express the probable error in each case as a fraction of the result. Thus a body is weighed and its mass is found to be 105.166 grams with a probable error of, let us say,  $\pm 0.00240$  gram. The probable error is in this case  $\pm 0.000024$  of the result, or 0.0024 per cent. of the result. A very small body is weighed and its mass is found to be 0.1326 gram with a probable error of  $\pm 0.0004$  gram. The probable error is in this case  $\pm 0.003$  of the result or 0.3 per cent. The probable error in the first weighing is six times as large as the probable error of the second weighing and yet the percentage error in the first case is nearly one hundred times as great as the percentage error in the second case.

**21. Fixed-condition measurements and varied-condition measurements.** Let us suppose that the acceleration of gravity  $g$  is to be determined by measuring the length  $l^*$  of a pendulum and the time  $t$  of one complete vibration of the pendulum,  $g$  being calculated by using the well-known formula

$$t = 2\pi\sqrt{\frac{l}{g}}$$

There are two widely different procedures which may be followed in making the measurements, namely, (a) A pendulum of *fixed length* may be used, and repeated measurements of  $l$  and  $t$  made; then from the mean of the measured values of  $l$  and the mean of the measured values of  $t$  the value of  $g$  may be calculated, and the probable error of the result may be estimated as explained in Arts. 16 and 17; or (b) The length  $l$  of the pendulum may be *varied*, and for each measured value of  $l$  the value of  $t$  may be observed. The procedure (a) may be called a *fixed-condition measurement*, and the procedure (b) may be called a *varied-condition measurement*.

\* If the pendulum consists of a metal ball of radius  $r$  hung by a fine wire or thread of length  $h$ , then  $l = h + r + \frac{2}{5} \frac{r^2}{h + r}$ .

**Adjustment or averaging of a set of varied-condition measurements.** The simple method of taking the mean of each measured quantity and calculating the result therefrom is not applicable\* in a varied-condition measurement, and the simplest method for adjusting (or averaging) such a set of measurements is by the graphical method as illustrated by the following examples:

*Example I.* The length  $l \left( = h + r + \frac{2}{5} \frac{r^2}{h + r} \right)$ , where  $h$  is the length of a fine suspending wire and  $r$  is the radius of a metal ball which is used as the pendulum bob) of a pendulum is varied, and for each measured value of  $l$  the time  $t$  of one vibration is observed. Then

$$t = 2\pi \sqrt{\frac{l}{g}}$$

Taking logarithms of both members of this equation, we get

$$\log t = \frac{1}{2} \log l + \log \frac{2\pi}{\sqrt{g}}$$

or

$$Y = \frac{1}{2}X + b$$

Plot on high-grade cross-section paper the logarithms of the observed values of  $t$  (the values of  $Y$ ) and the corresponding logarithms of the measured values of  $l$  (the values of  $X$ ). Except for errors of observation these plotted points will lie on a straight line.

Draw the best representative straight line through the plotted points.

Extend this straight line until it cuts the  $Y$ -axis, and measure or read off the intercept on the  $Y$ -axis. This gives the value of  $\log \frac{2\pi}{\sqrt{g}}$  from which the value of  $g$  may be calculated.

*Example II.* It is desired to find the focal length  $f$  of a simple converging lens. The lens is arranged to project the image of an object on a screen, and the distances,  $v$ , of the

\* Except when there is a linear relationship between the measured quantities. In the above example of the pendulum  $l$  is proportional to  $t^2$ , or, in other words, the relation between  $l$  and  $t$  is not linear.

image-screen from the lens are measured for a series of measured values of the distance  $u$  of the object from the lens. Then for any pair of measured values of  $u$  and  $v$  we have

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

or

$$\frac{1}{u} = -\frac{1}{v} + \frac{1}{f}$$

or

$$Y = -X + b$$

Plot the values of  $1/u$  (the values of  $Y$ ) and the corresponding values of  $1/v$  (the values of  $-X$ ). Except for errors of the measurements these plotted points will lie on a straight line.

Draw the best representative straight line through the plotted points.

Extend this straight line until it cuts the  $Y$ -axis, and measure or read off the intercept on the  $Y$ -axis. This gives the value of  $1/f$  from which the value of  $f$  may be calculated.

*General statement.* It is often possible to transform the equation which connects a calculated result and two observed quantities  $p$  and  $q$  so as to give

(a known function of  $p$ )

= (a known function of  $q$ ) + (a known function of  $R$ )

or

$$Y = X + b$$

Plot the pairs of "known function of  $p$ " ( $= Y$ ) and "known function of  $q$ " ( $= X$ ).

Draw the best representative straight line through the plotted points, and measure or read off the intercept on the  $Y$ -axis. This gives the value of the "known function of  $R$ " from which the value of  $R$  may be calculated.

*Remark.* The graphical method as above outlined does not enable one to estimate the probable error of a result.

**22. Empirical equations.** Consider the observed readings  $r$  of an ammeter which correspond to a series of known values of

the current  $i$ , or consider a set of observed values of any related quantities  $r$  and  $i$ . The relation between  $r$  and  $i$  can always\* be represented by a plotted curve, but it is often desirable to find an *algebraic formula* which expresses  $i$  in terms of  $r$ . Such a formula, when it is derived from the observed values of  $r$  and  $i$ , is called an *empirical equation*. The derivation of empirical equations is best explained by giving several typical examples as follows:

*Example I.* In some cases  $i$  can be expressed in terms of  $r$  with sufficient accuracy by an equation of the form

$$i = ar + b$$

where  $a$  and  $b$  are constants to be determined.

Plot the observed values of  $r$  and  $i$ .

If the plotted points lie very nearly on a straight line the equation  $i = ar + b$  is satisfactory, and to determine  $a$  and  $b$  proceed as follows:

Draw the best representative straight line through the plotted points, and extend this line until it cuts the  $i$ -axis.

The slope of the line, as determined from the plot, is the value of  $a$ , and

The intercept on the  $i$ -axis, as read off the plot, is the value of  $b$ .

*Example II.* Plot the observed values of  $r$  and  $i$ , and if a curved line without any point of inflection can be drawn so as to pass near to all of the plotted points, then it may be possible to represent  $i$  in terms of  $r$  with sufficient accuracy by an equation of the form

$$i = a + br + cr^2$$

where  $a$ ,  $b$  and  $c$  are constants to be determined as follows:

Plot the observed values of  $r$  and  $i$ , and draw the best representative smooth curve through the plotted points.

Choose three points on the smooth curve, one near the middle and one near each end.

\*We are not here concerned with those peculiar functions which cannot be plotted.

Read off  $i_1$  and  $r_1$  for the first point.

Read off  $i_2$  and  $r_2$  for the second point.

Read off  $i_3$  and  $r_3$  for the third point.

Substitute these three pairs of values in the above equation and we have

$$i_1 = a + br_1 + cr_1^2$$

$$i_2 = a + br_2 + cr_2^2$$

$$i_3 = a + br_3 + cr_3^2$$

Then calculate  $a$ ,  $b$  and  $c$  by means of these three equations.

To determine whether the resulting equation  $i = a + br + cr^2$  represents  $i$  to the required degree of accuracy the best thing to do is to plot the equation  $i = a + br + cr^2$ , using the values as above determined, and plot the observed values of  $i$  and  $r$  on the same sheet.

*Example III.* In some cases  $i$  can be expressed satisfactorily by an equation of the form

$$i = ar^b$$

where  $a$  and  $b$  are constants to be determined.

Taking logarithms of both members of this equation, we get

$$\log i = b \log r + \log a$$

or

$$Y = bX + c$$

Plot the pairs of values of  $\log i$  ( $= Y$ ) and  $\log r$  ( $= X$ ).

*If these plotted points lie very nearly on a straight line, then the form  $i = ar^b$  is satisfactory.*

Draw the best representative straight line through the plotted points and extend this straight line until it cuts the  $Y$ -axis.

The slope of this line, as determined from the plot, is the required value of  $b$ .

The intercept on the  $Y$ -axis, as read off the plot, is the value of  $\log a$ , from which the value of  $a$  may be found.

*Example IV.* In some cases  $i$  can be expressed satisfactorily by an equation of the form

$$i = ae^{br}$$

where  $e$  is the Naperian base, and  $a$  and  $b$  are constants to be determined.

Taking logarithms of both members of this equation, we get

$$\log i = (b \log e)r + \log a$$

or

$$Y = mX + c$$

Plot the pairs of values of  $\log i (= Y)$  and  $r (= X)$ .

*If these plotted points lie very nearly on a straight line, then the form  $i = ae^{br}$  is satisfactory.*

Draw the best representative straight line through the plotted points and extend this straight line until it cuts the  $Y$ -axis.

The slope of this line, as determined from the plot, is the value of  $b \log e$ , from which the value of  $b$  may be calculated.

The intercept on the  $Y$ -axis, as read off the plot, is the value of  $\log a$ , from which the value of  $a$  may be calculated.

*Example V.* In some cases the value of  $r$  can be expressed satisfactorily by an equation of the form

$$r = ae^{bi}$$

where  $e$  is the Naperian base, and  $a$  and  $b$  are constants to be determined.

Taking logarithms of both members of this equation we get

$$\log r = (b \log e)i + \log a$$

or

$$Y = mX + c$$

Plot the pairs of values of  $\log r (= Y)$  and  $i (= X)$ .

*If these plotted points lie very nearly on a straight line, then the form  $r = ae^{bi}$  is suitable; and the constants  $a$  and  $b$  are determined exactly as in Example IV.*

**23. Precision of graphical methods.** The abscissas and ordinates of a plotted curve cannot be laid down with greater accuracy than about  $\pm 0.02$  inch, and therefore, if we assume ordinates and abscissas to be, say, 4 inches long, the percentage inaccuracy is about one-half of one per cent. The graphical methods above outlined are not satisfactory when a greater precision than half-of-one-per cent. is required.

**24. Significant figures.** The volume of a rectangular parallelepiped is to be calculated from the measured dimensions which are as follows:

length	30.9 millimeters	$\pm 0.05$ millimeter.
breadth	15.7	$\pm 0.05$
height	10.5	$\pm 0.05$

The precise arithmetical result would be 5093.865 cubic millimeters; but the probable error of the result is about 30 cubic millimeters, and therefore the volume, as nearly as one would be justified in specifying it, is 5090 cubic millimeters. The last four figures or digits in 5093.865 are utterly meaningless.

**Rules for dropping insignificant figures or digits.** (a) *For addition.* Drop all digits in the result which fall under an unknown digit in any of the quantities to be added.

(b) *For subtraction.* Drop all digits in the result which fall under an unknown digit in  $A$  or under an unknown digit in  $B$ , where the result is  $A - B$ .

(c) *For multiplication.* Keep  $n$  significant figures or digits in the result, where  $n$  is the number of significant figures or digits in the least accurately known factor (The factors are the numbers which are to be multiplied together).

(d) *For division.* Consider the quotient  $A/B$  or  $B/A$ , and let  $n$  be the number of significant figures in  $A$ , where  $A$  is less accurately known than  $B$ . Then keep  $n$  significant figures in the result.

**25. Rules for approximation when calculating with small quantities.**—Another matter, which is on a par with the dropping of insignificant figures, in fact the two things are fundamentally identical, is the use of approximate formulas in calculating with small quantities. Most of the important cases fall under the general formula:

$$(1 + \delta)^m = 1 + m\delta$$

where  $m$  is any number whatever and  $\delta$  is small in comparison with unity. The particular cases most frequently encountered are:

$$(1 + \delta)^2 = 1 + 2\delta$$

$$(1 - \delta)^2 = 1 - 2\delta$$

$$\sqrt{1 + \delta} = 1 + \frac{1}{2}\delta$$

$$\sqrt{1 - \delta} = 1 - \frac{1}{2}\delta$$

$$\frac{1}{1 + \delta} = 1 - \delta$$

$$\frac{1}{1 - \delta} = 1 + \delta$$

*Example.* A clock pendulum of steel makes  $N = 24 \times 60 \times 60$  swings (half-oscillations) per day. How many more swings will it make per day if it is cooled one centigrade degree? The coefficient of linear expansion of steel is 0.000011 per centigrade degree. Therefore, if  $l$  is the length of the pendulum before it is cooled, its length after being cooled is  $l(1 - 0.000011)$ .

The number of complete oscillations per second of a pendulum is

$$n = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \quad (\text{i})$$

according to equation (8) on page 75 so that the number of complete vibrations per day is  $86400n$ , and the number of swings per day is

$$N = \frac{172800}{2\pi} \sqrt{\frac{g}{l}} \quad (\text{ii})$$

Let  $N + \Delta N$  be the number of swings per day made by the cooled pendulum; then

$$N + \Delta N = \frac{172800}{2\pi} \sqrt{\frac{g}{l(1 - 0.000011)}} \quad (\text{iii})$$

whence, dividing equation (iii) by equation (ii) member by member, we get

$$1 + \frac{\Delta N}{N} = \sqrt{\frac{1}{1 - 0.000011}} \quad (\text{iv})$$

But  $\sqrt{\frac{1}{1 - \alpha}} = (1 - \alpha)^{-1/2} = 1 + \frac{1}{2}\alpha$ , very nearly, so that

equation (iv) becomes

$$1 + \frac{\Delta N}{N} = 1 + 0.0000055$$

whence

$$\Delta N = 9.5$$

Let the reader calculate  $\Delta N$  from equation (iv) by ordinary arithmetic and he will appreciate the use of the approximation formula. To solve equation (iv) by logarithms requires the use of seven place tables, although the result ( $\Delta N$ ) is required to only two or three significant figures!

#### PROBLEMS.

**155.** The diameter of a steel rod is repeatedly measured and the following values were found 1.037, 1.028, 1.024, 1.033, 1.029, 1.030, 1.031, 1.029, 1.028 and 1.032 inches. Calculate the mean residual (which we will speak of as the probable error of a single measurement), calculate the probable error of the mean of the ten measurements, and calculate the probable error of the sectional area of the rod as calculated from the mean of the ten measurements.

**156.** Calculate the probable error of a single measurement of the diameter of the steel rod in the previous problem using the correct formula, and state why it is permissible to use the *mean residual* as a fairly satisfactory estimate of the *probable error* of a single measurement.

**157.** A result  $R$  is calculated from a measured quantity  $x$ , using the equation  $R = ax^n$ , where  $a$  and  $n$  are constants. Show that the percentage error in  $R$  is  $n$  times as large as the percentage error in  $x$ .

**158.** With what percentage error must the diameter of a rod be measured to give one per cent. error in the calculated sectional area of the rod?

**159.** With what percentage error must the diameter of a sphere be measured to give one per cent. error in the calculated volume?

**160.** A thermometer can be read with a probable error of  $\pm 0.1^\circ$ . What is the probable error of a temperature difference as read by the thermometer? What must be the value of the temperature difference in degrees in order that its probable error may be 0.4 per cent.?

**161.** The power delivered to a customer from direct-current

supply mains is measured by ammeter and voltmeter. The ammeter gives 79.2 amperes  $\pm 0.25$  ampere, and the voltmeter gives 109.6 volts  $\pm 0.74$  volt. Calculate the power, calculate its probable error in watts, and calculate its probable error in per cent.

**162.** The resistances of two coils are 16.24 ohms  $\pm 0.035$  ohm, and 10.26 ohms  $\pm 0.023$  ohm, respectively. Calculate the combined resistance  $R$  of the two coils when connected in series, and calculate the probable error of  $R$ . Calculate the combined resistance  $r$  of the two coils when connected in parallel, and calculate the probable error of  $r$ .

**163.** The diameter of a rod is 1.025 inch  $\pm 0.0025$  inch. What variation  $d\pi$  from the true value of  $\pi$  would introduce in the calculated sectional area of the rod an error  $3/10$  as large as the probable error in the calculated sectional area due to the probable error of the measured diameter?

**164.** What would the probable error in the measured diameter (approximately one inch) of a sphere have to be in order that one might use  $22/7$  for  $\pi$  in calculating the volume of the sphere, on the assumption that the error in  $V$  due to the error in  $\pi$  must not be larger, let us say, than  $2/10$  of the error in  $V$  due to the probable error in the measured diameter?

**165.** The volume of a slender steel rod is to be calculated from its measured diameter and length. The diameter is roughly  $1/4$  inch and it can be measured with a probable error of  $\pm 0.0005$  inch; and the length of the rod is roughly 3 feet. Find the probable error that is allowable in the measurement of the length of the rod if the error in the result due to error in measured length is to be, say,  $4/10$  of the error in the result due to the error in the measured diameter.

**166.** The restivity  $k$  of a sample of copper wire is determined by measuring the length  $L$  of the sample ( $= 256.2$  feet  $\pm dL$ ), the resistance  $R$  ( $= 2.665$  ohms  $\pm dR$ ), and the diameter  $D$  of the wire ( $= 0.0324$  inch  $\pm 0.0002$  inch). Find the errors  $dL$  and  $dR$  so that the component errors  $dK_L$  and  $dK_R$  may each be one fifth of  $dD$ .

167. A series of measured values of current, 10, 15, 20, 25, 30 and 35 amperes are passed through a wire which is kept at a constant temperature, and the corresponding values of the electromotive force across the terminals of the wire are measured and found to be 33.4, 50.6, 67.6, 87.5, 102.6 and 115.4 volts. Find the value of the resistance  $R$  of the wire by the graphical method.

168. A clock-pendulum of which the length is 24.9 centimeters makes half-second swings (172,800 swings in 24 hours). Calculate the change in length so that the pendulum will make 20 more swings per day.

169. A result  $i$  is calculated from an observed angle  $\theta$ , using the equation  $i = k \tan \theta$ , where  $k$  is a constant. The probable error of  $\theta$  is  $\pm 0.2^\circ$ . Find the value of  $\theta$  for which the percentage error of  $i$  is a minimum and find the value of this minimum percentage error. Ans.  $\theta = 45^\circ$ ; 0.7 per cent.

## APPENDIX C.

### CORRESPONDING EQUATIONS OF TRANSLATORY MOTION, ROTATORY MOTION AND "ELECTRICAL MOTION."

To every equation in translatable motion there is an exactly similar corresponding equation in rotatory motion and an exactly similar corresponding equation in "electrical motion" as exhibited in the following schedule. The converse of this statement is not true, that is to say, there are equations of rotatory motion which have no corresponding equations in translatable motion and there are equations of "electrical motion" and equations of rotatory motion which do not correspond.

$$x = vt \quad (1)$$

where  $x$  is the distance traveled in  $t$  seconds by a body which has a *constant* velocity  $v$

$$x = \frac{1}{2}at^2 \quad (4)$$

where  $x$  is the distance traveled in  $t$  seconds by a body which starts from rest and has a *constant* acceleration  $a$

$$W = Fx \quad (7)$$

where  $W$  is the work done by a force  $F$  when the body on which  $F$  acts moves distance  $x$  in the direction of  $F$ .

$$P = Fv \quad (10)$$

where  $P$  is the power developed by a force  $F$  when the body on which  $F$  acts moves at velocity  $v$  in the direction of  $F$ .

$$\phi = st \quad (2)$$

where  $\phi$  is the angle in radians turned in  $t$  seconds by a body which has a *constant* spin velocity of  $s$  radians per second.

$$\phi = \frac{1}{2}\alpha t^2 \quad (5)$$

where  $\phi$  is the angle turned in  $t$  seconds by a body which starts from rest and has a *constant* spin acceleration  $\alpha$

$$W = T\phi \quad (8)$$

where  $W$  is the work done by torque  $T$  when the body on which  $T$  acts turns through  $\phi$  radians about the axis of  $T$ .

$$P = Ts \quad (11)$$

where  $P$  is the power developed by a torque  $T$  when the body on which  $T$  acts turns about the axis of  $T$  at a speed of  $s$  radians per second.

$$q = it \quad (3)$$

where  $q$  is the amount of electric charge which flows in  $t$  seconds through a circuit in which a *constant* current  $i$  is flowing.

$$q = \frac{1}{2} \times \text{rate of growth} \times t^2 \quad (6)$$

If the current in a circuit grows at a constant rate starting at zero, the amount of charge which flows through the circuit in  $t$  seconds is equal to  $\frac{1}{2} \times \text{rate of growth of current} \times t^2$ .

$$W = Eq \quad (9)$$

where  $W$  is the work done by an electromotive force  $E$  when an electric charge  $q$  flows through the circuit on which  $E$  acts.

$$P = Ei \quad (12)$$

where  $P$  is the power developed by an electromotive force  $E$  when current  $i$  flows through the circuit on which  $E$  acts.

$$F = m \frac{dv}{dt} \quad (13)$$

where  $\frac{dv}{dt}$  is the acceleration produced by an unbalanced force  $F$  which acts on a body of mass  $m$ .

$$W = \frac{1}{2}mv^2 \quad (16)$$

where  $W$  is the kinetic energy of a body of mass  $m$  moving at velocity  $v$ .

$$F = ax \quad (19)$$

A ball is fixed to a spring and  $F$  is the force required to pull the ball to a distance  $x$  from its equilibrium position. The factor  $a$  is called the *stiffness coefficient* of the spring.

$$4\pi^2 n^2 m = a \quad (22)$$

A ball is fixed to a spring of which the stiffness coefficient is  $a$ . Then when the ball is pulled to one side and released it performs simple harmonic motion of which the number of complete vibrations per second is  $n$ ,  $m$  being the mass of the ball.

$$T = K \frac{ds}{dt} \quad (14)$$

where  $\frac{ds}{dt}$  is the spin acceleration produced by a torque  $T$  which acts on a wheel of which the spin-inertia is  $K$ .

$$W = \frac{1}{2}Ks^2 \quad (17)$$

where  $W$  is the kinetic energy of a wheel rotating at a speed of  $s$  radians per second, the spin-inertia of the wheel about its axis of spin being  $K$ .

$$T = b\phi \quad (20)$$

A body is hung by a wire and  $T$  is the torque required to turn the body (and twist the wire) through the angle of  $\phi$  radians. The factor  $b$  is called the *coefficient of torsional stiffness* of the wire.

$$4\pi^2 n^2 K = b \quad (23)$$

A body is hung by a wire of which the coefficient of torsional stiffness is  $b$ . Then when the body is turned so as to twist the wire and released it performs harmonic rotatory motion of which the number of complete vibrations per second is  $n$ ,  $K$  being the spin inertia of the body referred to the wire as an axis.

$$E = L \frac{di}{dt} \quad (15)$$

where  $\frac{di}{dt}$  is the rate of growth of current due to electromotive force  $E$  acting on a circuit of inductance  $L$ .

$$W = \frac{1}{2}Li^2 \quad (18)$$

where  $W$  is the kinetic energy of a current  $i$  in a circuit of which the inductance is  $L$ .

$$q = CE$$

or

$$E = \frac{1}{C} \cdot q \quad (21)$$

Two metal plates are separated by a layer of dielectric constituting what is called a *condenser*, and  $E$  is the electromotive force required to draw  $q$  coulombs out of one plate and force it into the other plate. The factor  $C$  is called the *capacity* of the condenser.

$$4\pi^2 n^2 L = \frac{1}{C} \quad (24)$$

A charged condenser of capacity  $C$  is connected to a circuit of inductance  $L$  and of negligible resistance. The discharge from the condenser then surges back and forth through the circuit, we have what is called an oscillatory discharge (harmonic), and the number of complete oscillations per second is  $n$ .

$$W = \frac{1}{2} \frac{1}{a} F^2 \quad (25)$$

$$W = \frac{1}{2} \frac{1}{e} T^2 \quad (26)$$

$$W = \frac{1}{2} C E^2 \quad (27)$$

A spring of which the stiffness coefficient is  $a$  is bent, the force which is acting on the fully bent spring is  $F$ , and the potential energy of the bent spring is  $W$ .

A wire of which the coefficient of torsional stiffness is  $b$  is twisted, the torque which is acting on the fully twisted wire is  $T$ , and the potential energy of the twisted wire is  $W$ .

A condenser of which the capacity is  $C$  is charged, the electromotive force which is acting on the fully charged condenser is  $E$ , and the potential energy of the charged condenser is  $W$ .

## APPENDIX D.

### ANSWERS TO PROBLEMS AND LEADING QUESTIONS.

On pages 5-7:

Problem 1. Ans.  $V = 123.2$  "pounds";  $H = 157.6$  "pounds."

" 3. Ans.  $T = 389$  "pounds";  $C = 298$  "pounds."

" 5. Ans. 5730 "pounds."

" 9. Ans. 2.95 miles per hour.

1. Give the exact definition of the "pound" as a unit of force, the London pull-pound.

2. Two men have a rope stretched between them and each man pulls on the rope with a force of 25 "pounds." What is the tension of the rope?

3. Three ropes are attached to a body at different points. Can the three forces exerted on the body by the three ropes be thought of as acting at the same point? If so, explain.

4. What is meant, physically, by the resultant of two forces?

5. A car is pulled by an inclined force  $F$  as shown in Fig. 10. Why is not this force  $F$  exactly equivalent to its component  $X$  in its effectiveness in moving the car?

6a. A man sits in a chair and pushes downwards on the chair. What is the reaction and on what body does the "reaction" act?

66. The *earth pulls downwards on a book* which lies on a table, and the *upward push of the table on the book* is **NOT** the reaction. What is the reaction? On what does the reaction act?

6c. A 160-pound man sits on a seat, braces his feet against a cleat and pulls on an oar. Specify verbally the forces which act on the man and state by what each force is exerted.

On pages 11-14:

7. Note. The "first condition of equilibrium" is the condition which must be satisfied in order that a set of forces may have no tendency to produce *translatory motion* (see Art. 21). If there are

more than three forces in the set it is not necessary that their lines of action should all intersect at the same point.

Problem 14, Ans.  $A = 123$  "pounds,"  $B = 96.8$  "pounds."

" 15, Ans. Case (b)  $F = 26.0$  "pounds."

" 16, Ans. Case (b) tension of rope = 126 "pounds."

" 17, Ans. 178.9 "pounds."

" 18, Ans. 1252 "pounds."

" 19, Ans. 175 "pounds."

" 21, Ans. Compression = 1005 "pounds."

7a. A book lies at rest on a table. What forces act on the book? If you were to take the table away what force would act on the book? How would the book behave?

7b. If every force is accompanied by an equal and opposite reaction, how is it that action and reaction do not always *balance* each other?

7c. A falling ball is acted on by a single force (friction of air being negligible), and, of course, this force is unbalanced; what is the reaction and on what is the reaction exerted?

7d. The push of a wall on a rebounding ball is an unbalanced, force, what is the reaction and on what is the reaction exerted?

8. Give a definition of the coefficient of friction of two sliding surfaces.

**On pages 18-20:**

*Note.* The "second condition of equilibrium" is the condition which must be satisfied in order that a set of forces may have no tendency to produce rotatory motion (see Art. 21).

Problem 22, Ans.  $F = 31.7$  "pounds."

" 24, Ans.  $F = 121$  "pounds."

" 25, Ans. 20 "pounds."

" 26, Ans. 274 "pounds."

" 27, Ans. 4620 "pounds."

" 28, Ans.  $W = 48$  "pounds."

" 29, Ans.  $A = 774$  "pounds."

9. What do you mean by the torque action of a given force about a given axis? In what terms is torque action expressed?

10. Imagine a man to be suspended in mid-air and entirely free

to move; what would happen to the man if he were to exert a torque on a screw by means of a screw-driver?

State the principle of equality of action and reaction as applied to torque.

11. Give an example of a set of forces which satisfy the first condition of equilibrium but do not satisfy the second condition.

12. Two forces, only, act on a long rod or beam and the forces are balanced. One of the forces acts at one end of the rod and the other force acts at the other end of the rod. Prove that both forces are parallel to the rod.

13. Prove that the lines of action of three forces must intersect at a point if the three forces are balanced.

On pages 22-25:

Problem 30, Ans.  $A = 430$  "pounds";  $B = 145$  "pounds."

" 31, Ans. Vert. comp. = 122.5 "pounds"; hor. comp. = 47.5 "pounds."

" 34, Ans. Tension of rope 3600 "pounds"; Vertical push of wall 200 "pounds" upwards; horizontal push of wall 3118 "pounds" to right.

" 35, Ans.  $F = 14.35$  "pounds";  $G = 18.6$  "pounds";  $H = 29.3$  "pounds."

" 36, Ans. Sise push of cross-head 44,000 "pounds"; compression of connecting rod 254,400 "pounds"; torque 390,000 "pound"-feet.

" 37, Ans. Tension of cable 3130 "pounds";  $R = L = 666$  "pounds."

" 38, Ans. Tension of cable 2861 "pounds";  $R = L = 666$  "pounds."

" 39, Ans.  $F = 40$  "pounds,"  $U = 115$  "pounds";  $V = 85$  "pounds."

On pages 27-28:

Problem 40, Ans.  $R = 450$  "pounds" 13.3 inches from  $O$ .

" 41, Ans.  $R = 50$  "pounds" 120 inches to right of  $O$ .

" 42, Ans.  $B = 5$  "pounds";  $Ob = 1.8$  feet.

**14a.** A certain force  $R$  is exactly equivalent in every respect to a given set of forces. What is the force  $R$  called? What is the relation between the component of  $R$  in any chosen direction and the sum of the components in that direction of all the forces of the set? Why? What is the relation between the torque action of  $R$  about any chosen axis and the combined torque action about that axis of all the forces of the set? Why?

**14b.** A set of forces, unbalanced, act on a table. In what three respects is the complete resultant of the set equivalent to the set? In answering this question it is helpful to consider the table as capable of three kinds of motion, namely, (a) Motion to east or west, (b) Motion to north or south, and (c) Rotatory motion about any vertical axis.

**On pages 33-34:**

Problem 43, Ans. 8950 joules; 814 watts.

" 44, Ans. 597 watts.

" 45, Ans. 132,000 "pounds."

" 46, Ans. 4.66 cents.

" 47, Ans. 0.836 cent.

" 48, Ans. 1.39 cents.

" 50, Ans. 1751 "pound"-feet. See Appendix C.

" 51, Ans. 18.8 horse-power.

**15.** Can a balanced force be an "active" force, that is, can a balanced force do work? Give an example.

**16.** Give an example of an "inactive" unbalanced force, a force which is accelerating a body, but which does not do work.

**17.** A cart moves due northwards on a level road at a velocity of 2.75 feet per second. A man pushes straight downwards on the cart with a force of 200 "pounds" and a mule pulls due northwards on the cart with a force of 100 "pounds." Find the rate at which the man does work and the rate at which the mule does work.

**18.** The *erg* is the c.g.s. unit of work and it is the work done by a force of one dyne (see Art. 29) while the body on which it acts moves one centimeter in its direction. What is a joule? What is a foot-"pound" of work?

19. Does a horse power have money value? Of what is a horse-power-hour a unit? How much is it?

20. Prove that the power developed by an active force is equal to the product of the force by the velocity with which the body on which the force acts moves in the direction of the force.

21. The force required to propel a boat is, let us say, proportional to the velocity of the boat. If one horse power is required to propel the boat at a velocity of 2 miles per hour how much power would be required to propel the boat at a velocity of 4 miles per hour? Explain.

22. What is meant by the statement that a body stores energy? Can a car load of stone do work? If so, how?

23. What is meant by potential energy?

24. What is meant by kinetic energy? See Art. 37 on pages 60 and 61.

25. What becomes of the work which is done in lifting a body? A force acts on a body and sets the body in motion (no friction); what becomes of the work done by the force?

What becomes of the work which is done in bending a spring?

What becomes of the work done in overcoming friction as in dragging a box along the floor?

**On pages 36-37:**

Problem 53, Ans. 29.5% utilized.

" 55, Ans. 785 "pounds."

**On page 48:**

Problem 67c, Ans.  $k = 5 \times 10^{-6}$  dollars per day *per day per day*; \$1667.

26. Can a body be at a certain place and have a velocity?

Certainly, but it will not remain at that place.

Can a body have zero velocity and yet have an acceleration?

Give an example.

Can a changing quantity be equal to zero?

Can a man without a bank account be saving money?

**On pages 53-54:**

Problem 58, Ans. 0.667 feet per second per second; draw-bar pull = 18,600 "pounds."

Problem 59, Ans. 14,800 "pounds." Locomotive truck wheels are assumed to be frictionless.

" 60, Ans. 26,270 "pounds."

Problem 60b. An 80-pound block and a 60-pound block lie in contact with each other on a smooth (frictionless) table. A steady horizontal force  $F$  is exerted on the 60-pound block pushing it towards the 80-pound block. Find  $F$  in order that the 60-pound block may exert a push of 10 "pounds" on the 80-pound block.

Problem 61, Ans. 0.215 feet per second per second.

" 62, Ans. (a) 1880 "pounds"; (b) 2500 "pounds."

" 63, Ans. (a) 12.48 "pounds"; (b) 7.52 "pounds."

Problem 64, Ans. 989.0 "pounds."

" 65, Ans. 2.93 feet per second per second. Tension = 10.91 "pounds."

" 66, Ans. 9.66 feet per second per second. Tension = 10.5 "pounds."

27. Can one body travel around another and have always pure translatory motion (no rotation)?

28. A slender uniform stick is held in a vertical position in one hand (between thumb and forefinger) and the hand is moved suddenly sidewise exerting a horizontal force on the stick. How does the stick move if the thumb-and-finger-hold is above the middle of the stick? How does the stick move if the thumb-and-finger-hold is at the middle of the stick? Illustrate by diagrams.

29a. Is it sufficient to define the center of mass of a body as the point at which all of the material of the body may be thought of as concentrated?

29b. Define what is meant by the average velocity of a body during a given time? By the average acceleration? Give an example illustrating precisely what is meant by the velocity of a body at a given instant. By the acceleration of a body at a given instant.

29c. Newton's first law refers to the behavior of a body upon which no force acts or on which the forces which do act are balanced. What is the behavior?

**29d.** State Newton's second law of motion?

Is it correct to say that the effect of an unbalanced force is to *move* a body?

**29e.** Newton's third law of motion refers to the relation between action and reaction. State the law.

**30.** The pull of the earth on a one-pound body in London accelerates the one-pound body at the rate of 32.174 feet per second per second. By what principle or law do you know that *the same amount of force* (the London pull-pound) would accelerate a 32.174-pound body at the rate of one foot per second per second? State the principle.

**31.** A certain force accelerates a given body at the rate of 5 feet per second per second. By what principle or law do you know that a force twice as great would accelerate the given body at the rate of 10 feet per second per second? State the principle.

**32.** What is a mass of 32.174 pounds (sugar-pounds) called? Why is it a convenient unit of mass?

**33.** Define the c.g.s. unit of force.

**34.** "Weighing" a body on a balance scale does not give what is technically called the weight of the body, but its mass in pounds. How could you determine the true weight of a body in London pull-pounds at any place where the acceleration of gravity is known, if you know the mass of the body in sugar-pounds, having "weighed" it on a balance scale? Explain.

**35.** The mass of a body in slugs is certainly equal to  $W/g$ , where  $W$  is the true weight of the body *at any given place* in London pull-pounds and  $g$  is the acceleration of gravity at that place. A coal dealer sends you a batch of coal and in his bill he states that the "weight" of the coal is 2000 pounds. Is it correct to use 2000 pounds for  $W$  in the above formula and divide by the local value of  $g$  to get the mass of the coal? Is it merely a question of numerical precision? How about dividing pounds of coal (as measured by a balance scale) by a *denominate number* to get slugs?

**On pages 59-69:**

Problem 67, Ans. 462 feet; 157 feet.

Problem 69, Ans. Horizontal distance 12.99 feet; vertical distance 1.55 feet below starting point.

“ 70, Ans. (a)  $1/192$  second; (b) 3072 feet per second per second; (c) 518 “pounds.”

“ 71, Ans. (a) 0.01 second; (b) 240,000 feet per second per second; (c) 1,500,000 “pounds.”

Problem 71*b*. A passenger train if it did not stop at a station would pass by at a uniform speed of, let us say, 40 miles per hour. How much time is lost in making a 30-second stop at the station on the assumption that the train will have a uniform deceleration of 2 feet per second per second while stopping, and a uniform acceleration of 1.75 feet per second per second while getting up to full speed again?  
Ans. 61.42 seconds.

Problem 73, Ans. 39.3%; 28.4%.

On pages 63-64:

“ 75, Ans. 26 feet per second.

“ 76, Ans. Recoil velocity 25.4 feet per second; kinetic energy of gun  $2.92 \times 10^8$  foot-“pounds”; kinetic energy of projectile  $1.89 \times 10^8$  foot-“pounds”; kinetic energy of powder gases due to forward motion  $2.80 \times 10^8$  foot-“pounds.”

Problem 77, Ans. 32,800 centimeters per second.

*Note.* Calculate from the given data, how far the pendulum body is lifted; and from this calculate the velocity imparted to the pendulum body by the bullet.

Problem 78, Ans.  $P = 3661$  “pounds.”

“ 79, Ans.  $P = 2250$  “pounds.”

“ 80, Ans. (a) Upward push on front wheels 714.3 “pounds,” on back wheels 885.7 “pounds”; (b) Upward push on front wheels 1028.6 “pounds,” on back wheels 571.4 “pounds.”

Problem 80*b*. The legs of a chair touch the level floor of a car at the corners of a 16-inch square and one edge of the

square is parallel to the direction of acceleration of the car. Find the acceleration which is just enough to tip the chair over, center of mass of chair and its load being 24 inches above the center of the above mentioned square. Ans. 10.7 feet per second per second.

36. A bullet moving at a velocity of 1000 feet per second strikes and buries itself in a block which is free to move and whose mass is 999 times as great as the mass of the bullet. What is the velocity of the block and bullet after impact? What is it that has been *conserved* in this case? What would the velocity of the block and bullet have to be to have the same kinetic energy as the bullet had when moving at the velocity of 1000 feet per second? What fraction of the original kinetic energy remains after the impact as the kinetic energy of bullet and block? What has become of the remainder?

37. From the principle of action and reaction as stated in Art. 5 on page 5 prove that the mutual force action of two bodies must always produce equal changes of momentum in the two bodies and in opposite directions.

38. Show that an unbalanced force which acts on a body is equal to the rate of change of momentum of the body and parallel thereto.

39. The "impulse value" of a force is equal to the average value of the force multiplied by the time that the force continues to act. "Impulse values" are often employed in discussing impact. Show that the "impulse value" of a force is equal to the momentum it can produce.

40. In terms of what unit is momentum expressed (a) In the c.g.s. system; (b) In the f.s.s. system?

41. Watching a standing automobile you see it shoot suddenly forwards. What thing exerted the necessary forward push on the automobile? At what point on the automobile structure was this forward push applied?

42. Imagine an automobile to "float" in space with no force whatever acting upon it, and imagine the drive wheels to be locked. (a) How would the automobile move (what kinds of

motion would be produced) if a forward force were applied to the rims of the drive wheels at the points where the rims would normally touch the ground?

When an automobile is started on a level road the road bed pushes forwards on the rims of the drive wheels. (b) Why is it that the automobile does not move as in (a)? Answer this question by stating what new force or forces "do the business," state directions and points of application of these new forces, and use a diagram.

43. How do you know that all the forces which act on a starting automobile are together *equivalent* to a single forward force acting at or through the center of mass of the automobile? What does the word *equivalent* mean?

44. All the forces which act on the elevator cage in Fig. 76 are together equivalent to a single downward force acting at the center of mass  $C$  when the car is starting downwards. Is the "single downward force" here referred to the same thing as the downward force of 3000 "pounds" which is shown in Fig. 76? If not, why not?

45. Why is it that the combined torque action of all the forces which act on an automobile which is starting on a level road is zero about any horizontal axis at right angles to the road and  $h$  feet above the road where  $h$  is the height of the center of mass of the automobile above the road? See question 43 as to the meaning of the word *equivalent*.

**On pages 70-71:**

Problem 82a, Ans. 35,000 "pounds."

" 82b, Ans. 1.03 feet.

" 83, Ans. (a) 3.43 centimeters per second per second;  
(b) 982.5 dynes.

" 84, Ans. 0.272 dynes.

" 85, Ans. One revolution in 5.27 seconds.

" 87a, Ans. 1600 "pounds."

" 87b, Ans. 21.5 "pounds."

Problem 87c, Ans. 89.6 feet per second.

" 87d, Ans. 20,300 "pounds."

Problem 87e, Ans. 1210 revolutions per minute.

Problem 87f. A body whose velocity at a given instant is 90 feet per second has an acceleration of 25 feet per second in a direction which is inclined at an angle of  $30^\circ$  to the direction in which the body is traveling. Find the radius of curvature of the path at the given place and find the rate at which the velocity of the ball is increasing in value. Ans. Radius of curvature of path 648 feet; rate of increase of value of velocity 21.65 feet per second per second.

46. When a ball is twirled in a circle on the end of a string is its motion merely *translatory motion in a circle* and nothing more?

47. Take hold of a long slender stick and give it pure translatory motion in a circle. At what point may the entire mass of the stick be thought of as concentrated during this translatory motion in a circle?

48. A particle travels along a curved path and the acceleration of the particle at any instant may be resolved into two components, namely, (a) a component parallel to the direction of the path at the point where the ball is at the given instant (parallel to the direction in which the ball is traveling) and (b) a component at right angles to the direction in which the ball is traveling. What is the effect of the component (a) of the acceleration? What is the effect of the component (b) of the acceleration? What is the relation between the velocity  $v$  of the particle at the given instant, the radius of curvature  $r$  of its path and the sidewise component (b) of its acceleration?

49. What is meant by the nosing of a locomotive? What is an easement curve?

50. A bit of cloth to which are clinging innumerable droplets of water is placed in a rapidly rotating bowl with perforated sides. Consider a particle or drop of water. What force must act on this drop as the bowl rotates? This force is, of course, exerted on the particle of water by the cloth because nothing else touches the particle of water; the cloth *adheres* to the water (or the water *adheres* to the cloth). What happens when the force which would be needed to hold the particle of water in its circular

path is greater than the force with which the cloth adheres to the water?

On page 76:

Problem 88*a*, Ans. 197.3 feet per second per second; 49.8 "pounds."

" 88*b*, Ans. 18.85 feet per second; 22.2 foot-"pounds."

" 88*c*, Ans. Kinetic energy 16.7 foot-"pounds"; potential energy 5.5 "foot-pounds."

" 89, Ans. 7.12. periods per second.

" 91, Ans. 24.9 centimeters.

51. The acceleration  $a$  of a particle in harmonic motion of frequency  $n$  is  $a = -4\pi^2 n^2 x$ , according to an unnumbered equation near the bottom of page 74, where  $x$  is the distance of the particle (at the given instant) from its position of equilibrium as indicated in Fig. 87. Putting the time of one complete vibration,  $t$ , for  $1/n$ , and neglecting the negative sign, we get

$$t = 2\pi \sqrt{\frac{x}{a}}$$

This equation is, of course, exactly equivalent numerically to the equation  $a = -4\pi^2 n^2 x$ , but it is a purely kinematical equation. It does not express the dynamics of harmonic motion. To correlate harmonic motion with the physical conditions under which it takes place the three equations (5), (6) and (7) on pages 72, 74 and 75, or equivalent equations, must be used.

52. Under what conditions does a particle perform harmonic motion? What is meant by the "stiffness coefficient" of a spring?

53. An ideal simple pendulum consists of a small ball or bob of mass  $m$  supported by a rod or thread of length  $l$  and of which the mass is negligible. The bob is displaced sidewise through a very small distance  $x$ ; show that the force  $F$  which tends to bring the bob back is proportional to  $x$  and derive an expression for the proportionality factor or "stiffness coefficient" and explain precisely what units are used throughout; then derive the expression for the time of one complete vibration of the pendulum.

**On pages 83-84:**

Problem 92, Ans. (a) 554,600 gram-(centimeters)<sup>2</sup>; (b) 550,000 gram-(centimeters)<sup>2</sup>.

“ 93, Ans. (a) 370 seconds; (b) 0.34 radians per second per second; (c) 8.5 “pound”-feet of torque.

“ 95, Ans. 28.9 centimeters.

“ 96, Ans. 57.7 centimeters.

“ 97, Ans.  $6.11 \times 10^7$  gram-(centimeters)<sup>2</sup>; 30.7 centimeters.

54. Define what is meant by the average spin velocity of a wheel during a given interval of time, and also what is meant by the average spin-acceleration of a wheel during a given interval of time. Let  $\phi = at^4$  be the angle in radians turned through by a wheel in  $t$  seconds. Find expressions for instantaneous spin-velocity and instantaneous spin-acceleration.

55. Is it true in general that a body on which no torque acts, or on which the torques which do act are balanced, will continue to rotate at an unchanging spin-velocity about a fixed axis? If not, describe and explain a case in illustration.

56. A given body has, of course, a definite mass. Does a given body have a definite moment of inertia?

57. Consider a wheel whose spin-velocity is  $s$ . (a) Find an expression for the velocity of a particle of the wheel whose distance from the axis of rotation is  $r$ . (b) Find an expression for the kinetic energy of the particle of which the mass is  $\Delta m$ . (c) Formulate an expression (as a sum or integral) for the total kinetic energy  $W$  of the rotating wheel. (d) If the moment of inertia  $K$  of the wheel referred to its axis of rotation be defined on the basis of the equation  $W = \frac{1}{2}Ks^2$ , show that  $K = \sum r^2 \cdot \Delta m$ .

58. Consider a wheel whose spin-acceleration (about a fixed axis) is  $\alpha$ . (a) Derive an equation for the sidewise acceleration of a particle of the wheel at a distance  $r$  from the axis, by sidewise acceleration is meant “at right angles to  $r$ .” (b) Derive an expression for the sidewise force  $\Delta F$  which must be acting on the particle,  $\Delta m$  being the mass of the particle. (c) Write

down an expression for the torque action  $\Delta T$  of the force  $\Delta F$  about the axis of the wheel. (d) Derive an expression (as a sum or integral) for the total torque  $T$  which must be acting on the wheel to produce the specified spin-acceleration  $\alpha$ . (e) Using equation (9) on page 78 as a basis for the definition of  $K$ , show that  $K = \Sigma r^2 \cdot \Delta m$ .

59. Differentiate the equation  $W = \frac{1}{2}K\alpha^2$  with respect to time, and using the equation  $P = Ts$  (see Appendix C), derive equation (9) on page 78. Explain the physical meaning of each step. In the same way derive the equation  $F = ma$  from  $W = \frac{1}{2}mv^2$  and explain the physical meaning of each step.

60. Derive an expression for the moment of inertia of a long slender rod (uniform) referred to an axis at right angles to the rod and passing through the middle of the rod. Use calculus in this derivation.

61. What is the c.g.s. unit of moment of inertia? What is the f.s.s. unit?

62. What is meant by the radius of gyration of a body? A sphere is attached to an arm so as to rotate around an axis like one of the balls in Fig. 90. Is the distance from the axis to the center of mass of the sphere its radius of gyration? Explain.

### On pages 88-90:

Problem 98, Ans. 350 centimeters per second per second.

*Note.* In formulating this problem (and in formulating the problem which is discussed in Art. 55) why is it necessary to reckon the torque action of all the forces about *an axis passing through the center of mass* of the rolling body and place this torque equal to  $K\alpha$ , where  $K$  is the spin inertia of the body about that axis and  $\alpha$  is the spin acceleration? The answer is that it is not necessary, but it is by far the simplest procedure. This matter may be explained as follows: In the first place any set of forces is equivalent either to a pure torque or to a single force acting at some point of the body as explained in Art. 13. A pure torque always sets a body in simple rotation about an axis passing through the center of mass of the body and this case need not be considered. It remains, therefore, to consider the effect of a single force not passing through the center of mass of the body as indicated in Fig. 176. The motion of the body  $BB$  may be formulated as explained in Art. 55, namely

$$F = ma \quad (i)$$

$$Fl = K\alpha \quad (ii)$$

where  $a$  is the translatory acceleration of the body  $BB$  (the acceleration of its

center of mass  $C$ ),  $m$  is the mass of the body,  $K$  is the spin inertia of the body about an axis through  $C$ , and  $\alpha$  is the spin acceleration of the body about the axis through  $C$ .

Or, the motion of the body  $BB$  may be formulated on the basis of equation (i) and a second equation expressing the fact that the rate of gain of "moment of momentum" about an axis through  $p$  is zero. The "moment of momentum" of the body about an axis passing through  $p$  is made up of two parts, namely, (a)  $Ks$ , where  $K$  is the spin inertia about an axis passing through  $C$  and  $s$  is the spin velocity, and (b)  $-lmv$ , where  $l$  is the distance  $pC$ ,  $m$  is the mass of the body and  $v$  is the velocity of  $C$  to the right. Therefore the total "moment of momentum" about an axis through  $p$  is  $Ks - lm v$ , and its rate of change is  $K\alpha - lma$ . Therefore we have  $K\alpha - lma = 0$ .

Velocity and acceleration are considered positive towards the right; spin velocity and spin acceleration are considered as positive when clock-wise.

But  $ma = F$ , so that  $lma = Fl$ ; and consequently the equation  $K\alpha - lma = 0$  is exactly equivalent to equation (ii) above. Therefore we might just as well reckon the torque action of  $F$  about an axis passing through  $C$  and set up equation (ii) without bothering with the rather complicated general idea of "moment of momentum."

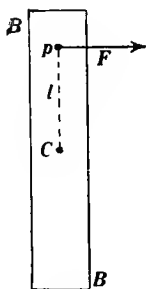


FIG. 176

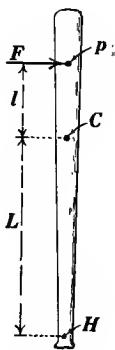


Fig. 177

**The problem of the ball bat.** Figure 177 shows a ball bat which is rotating about a point  $H$ , and it is required to find the point  $p$  where a *single force* (no side force to act on the bat at  $H$ ) must be exerted on the bat to bring it to rest, or, what amounts to the same thing, it is required to find the point  $p$  where a *single force*  $F$  must be applied to set the bat rotating about  $H$ .

The translatory acceleration  $a$  of the bat (the acceleration of the center of mass  $C$ ) satisfies the equation

$$F = ma \quad (i)$$

where  $m$  is the mass of the bat.

The torque action of  $F$  about an axis through  $C$  is  $Fl$ , and this torque action produces a certain spin acceleration  $\alpha$  of the bat so that

$$Fl = K\alpha \quad (ii)$$

where  $K$  is the spin inertia of the bat about an axis through  $C$ .

The point  $H$  has translatory acceleration  $a$  towards the right because of the translatory acceleration  $a$  of the bat as a whole, and the point  $H$  has translatory acceleration  $L\alpha$  towards the left on account of the spin acceleration  $\alpha$  of the bat about an axis through  $C$ . Therefore, if the point  $H$  is to have zero acceleration we must have

$$L\alpha = a \quad (\text{iii})$$

and from equations (i), (ii) and (iii) we get

$$l = \frac{K}{mL} \quad (\text{iv})$$

The point  $p$  is called the *center of percussion* of the bat with respect to the point  $H$ .

A very interesting application of the problem of the ball bat is as follows: An ordinary table knife is supported on the finger at  $H$ , Fig. 178, and the end  $p$

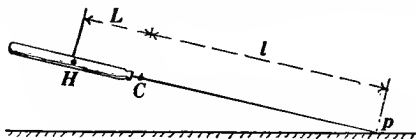


Fig. 178.

of the knife falls and strikes the table. If equation (iv) is satisfied, that is if  $p$  is the center of percussion of the knife with respect to  $H$ , the knife will make a large number of vibrations (hitting the table and rebounding) before its energy is dissipated; but if equation (iv) is not satisfied, then a side force\* must be exerted at  $H$  by the finger, the finger will be moved by the reacting force with which the knife pushes on the finger, a very considerable amount of work will thus be done on the finger, the energy of the knife will be quickly dissipated and the knife will very quickly come to rest. Try the experiment for yourself.

\* This does not refer to the *steady* force which must be exerted by the finger.

Problem 99, Ans. 980 centimeters per second per second.

“ 100, Ans. 19.4 centimeters per second per second;  
 $9.7 \times 10^6$  dynes.

“ 101, Ans. (a) 0.450 feet per second per second;  
 (b) 0.90 feet per second per second.

“ 103, Ans. 1925 dyne-centimeters per radian.

63. Explain how to determine the ratio of translational kinetic energy and rotational kinetic energy of a rolling body when  $m$  and  $K$  and the radius of the rolling circle are given.

64. Derive an expression for the time of one complete vibration of a physical pendulum, and explain the physical meaning of each step.

On pages 93-94:

Problem 106, Ans. 882,000 dyne-centimeters; 3.02 seconds.

Problem 107, Ans. 229,600 slug-(feet)<sup>2</sup>.

" 108, Ans. 51.4 "pounds."

" 109, Ans. 1030 "pounds."

It seems strange that, to get the value of the torque  $T$  in Figs. 104 and 105, the spin-acceleration  $\Delta s/\Delta t$ , *which is about an axis parallel to  $T$* , must be multiplied by the spin-inertia  $K$  of the wheel referred to its axis of spin, *which is at right angles to  $T$* . This curious absurdity is avoided by basing the discussion of Figs. 104 and 105 on the idea or notion of spin-momentum as follows:

The product  $Ks$  is called the *spin-momentum* of a spinning wheel, and an unbalanced torque is always equal to the rate of change of spin-momentum. At a certain instant the spin-momentum of the wheel in Figs. 104 and 105 is  $Ks$ , and  $\Delta t$  seconds later its spin-momentum is  $Ks'$ , *and in both expressions  $K$  is evidently the spin-inertia of the wheel referred to its axis of spin (the axle in Fig. 104).*

The vector difference  $Ks' - Ks$  is the change of spin-momentum during the time  $\Delta t$  and it is equal to  $K \cdot \Delta s$  and parallel to  $\Delta s$ . Therefore, dividing  $K \cdot \Delta s$  by  $\Delta t$  we get the rate of change of the spin-momentum,  $K \cdot \Delta s/\Delta t$ , which is equal to the torque  $T$ .

**On pages 98-99:**

Problem 112, Ans. 5.33 inches.

65. Do the terms *pressure* and *force* have the same meaning? If they do, why the two words? If they do not, what, exactly, is the meaning of pressure?

66. Is it sense or non-sense to say that the pressure of or in a fluid is the same in all directions?

67. State Pascal's principle.

68. The force  $dF$  which is exerted on a small plane area  $da$  by a fluid at rest is  $dF = p \cdot da$ . This equation is always true whether the pressure  $p$  is uniform or non-uniform. Ask your professor of mathematics to define  $dF$  and  $da$  as finite differentials in accordance with his hobby relative thereto. It can be done, but no professor of mathematics can do it so that the

result of his definition can be intelligibly used by anybody, not even by himself.

**On pages 103-104:**

Problem 115, Ans. 288 feet.

Problem 115*b*. A U-tube is partly filled with mercury (specific gravity 13.6), one arm of the tube is open to the air and the other arm connects to (communicates with) the outlet box or chamber of a fan blower. The mercury stands 7.5 inches higher in the open arm than in the other arm. Find the gage pressure in the outlet box in "pounds" per square inch. Gage pressure means pressure above atmosphere. Ans. 3.68 "pounds" per square inch.

Problem 115*c*. A U-tube is partly filled with mercury and the remainder of the tube is filled with water. The arms of the tube communicate two vessels *A* and *B* which are full of water (vessels *A* and *B* are at the same level). The mercury in one arm of the U-tube stands 7.5 inches higher than in the other arm. Find the difference in pressure in *A* and *B*. Ans. 3.42 "pounds" per square inch.

Problem 116, Ans. 26.9 "pounds" per square inch.

" 117*a*, Ans. 57,600 "pounds."

" 117*b*, Ans. 8,590 "pounds."

" 118, Ans. 22,920 "pounds."

Problem 118*b*. The vertical side of a tank has in it a triangular hole. The apex of the hole is at water level and the bottom edge of the hole is horizontal and 6 feet wide and 10 feet below water level. The hole is closed by a closely fitting triangular board. Find the total push of the water on the board, and find how far the "center of pressure" is below the water level. The center of pressure is the point of application of the single force which is the resultant of (is equivalent to) all the forces exerted on the various parts of the board by the water. Ans. 12,500 "pounds"; 7.5 feet.

69. Prove that the pressure would be everywhere the same in a body of fluid at rest if the fluid were not acted upon by gravity?

70. In deriving an expression for  $p - p_0$  on page 100 why

is it permissible to consider the force exerted on the *horizontal* area  $a$  in Fig. 118? How do you know that the force exerted on this area by the liquid is equal to the weight of volume  $ax$  of the liquid? How do you know that the true weight in London pull-pounds of this volume of liquid is  $axdg$ , where  $d$  is the density of the liquid in slugs per cubic foot?

71. Consider a triangular hole in the vertical side of a tank, the apex of the hole being at water level and the lower edge of the hole being horizontal,  $D$  feet below water level and  $b$  feet wide. The hole is closed by a triangular board which fits it accurately and it is required to find the "center of pressure" of this triangular board, that is, to find how far the "center of pressure" is below the water level, the "center of pressure" being the point of application of the complete resultant of all the forces exerted by the water on the board. See Art. 13. (a) How wide is the board at distance  $x$  beneath water level? (b) What is the area of the narrow strip between  $x$  and  $x + dx$ ? (c) What is the force  $\Delta F$  exerted on this strip? (d) What is the total force  $F$  exerted on the whole board? (e) What is the torque action of  $\Delta F$  about an axis at water level (axis parallel to face of board)? (f) What is the total torque action  $T$  about this axis of all the forces exerted on the board? (g) How far below water level must the total force  $F$  be to produce the torque  $T$  about the specified axis?

On pages 109-110:

Problem 121, Ans. 335 "kilograms."

" 122, Ans. 785,500 cubic feet; 16 inches.

" 123, Ans. 1.751 grams per cubic centimeter.

" 124, Ans. 102 grams.

" 125a, Ans. 32.1 pounds.

" 125b, Ans. 22.54 centimeters.

72. State Archimedes' principle for a totally submerged body. For a floating body.

73. Is it correct to say that the density of water is unity in the c.g.s. system? Is it correct to specify the density of oil as 7 pounds per gallon?

74. What does it mean to say that the specific gravity of a given substance is 6.263 at  $15^{\circ}\text{C}.$ ?

75. Why does a thin layer of lard tend to heap up on the cooler portions of the bottom of a frying pan?

**On pages 120-121:**

Problem 129, Ans. 940,000 foot-"pounds."

76. What is meant by a permanent state of flow? By a varying state of flow? Give examples of each. What is meant by simple flow (permanent state)? What is meant by a stream line?

77. What is meant by lamellar flow? Give a detailed example of non-lamellar flow using a diagram.

78. A rigid (non-expansible) pipe is solid-full of flowing water. What can be said of the density of the water in the pipe if the product  $av$  at one end of the pipe is not equal to the product  $a'v'$  at the other end of the pipe, where  $a$  and  $a'$  are the sectional areas of the pipe and  $v$  and  $v'$  are the velocities (assumed to be lamellar) of the water at the respective ends of the pipe?

79. In what terms is energy per unit volume expressed? How can energy per unit volume be equal to pressure?

**On pages 128-130:**

Problem 131, Ans. 100.6 feet per second.

Problem 131b. Taking the density of air as  $1/800$  of the density of water, find the number of foot-"pounds" per second of power represented by the kinetic energy of the air which blows through a 16-foot circle (16 feet in diameter) in a wind which blows at a velocity of 16 miles per hour. Government tests made by E. C. Murphy in 1898 show that a 16-foot wind mill develops 0.90 horse power in a 16-mile wind. Find the efficiency of the wind mill. Ans. 3170 foot-"pounds" per second; 15.6 per cent.

Problem 131c. Assuming the efficiency to be constant, how does the power of a wind mill vary with the velocity of the wind? Why?

Problem 131d. A fan blower delivers 25 cubic feet of air per second and cylindrical pipe outlet of the blower is 8 inches

in diameter. A U-tube partly filled with water is connected *to the side* of this cylindrical pipe (other arm of U-tube being open to the air) and the water stands 3.2 inches higher in the open arm. Calculate the total amount of power represented by the air stream as delivered by the fan blower, neglecting the slight effect of compressibility of the air. If the fan blower has an efficiency of 75 per cent., and if it is driven by an electric motor whose efficiency is 80 per cent., how many kilowatts of electrical power is taken from the supply mains? Ans. 578 foot-"pounds" per second; 1.31 kilowatts.

Problem 133, Ans. 0.404 "pound" per square inch.

" 134, Ans. 83.9 "pounds" per square inch.

" 135, Ans. 5.8 cubic feet per second.

" 137, Ans. 4 feet per second.

" 138, Ans. 101.6 feet per second.

Problem 138b. A Pitot tube is mounted on top of a building so as to be exposed to the wind, and the tube is kept in the proper position with reference to the wind (see Fig. 141 on page 127) by a weather vane. Two long air-tight tubes lead to a convenient place in the building where the pressure difference developed by the wind is balanced by water in a U-tube as indicated in Fig. 141. Calculate the value of  $h$ , Fig. 141, for wind velocities of 10, 20, 40 and 80 miles per hour, taking the density of the air to be  $1/800$  of the density of water. Ans. 0.05 inch; 0.20 inch; 0.80 inch; 3.2 inches.

Problem 138c. A Pitot tube is mounted on an air plane so as to be exposed (as indicated in Fig. 141) to the apparent wind which is due to the motion of the air plane. The U-tube contains colored water and the difference of level  $h$  is observed to be 5.5 inches. What is the velocity of the air plane through (relative to) the air? Calculate the result for two altitudes: (a) At sea level where the density of the air is about  $1/800$  of the density of water, and (b) At a high altitude (about 10,000 feet above sea level) where the density of the air is about  $1/1200$  of the density of water. Ans. (a) 104.4 miles per hour; (b) 128.1 miles per hour.

80. What is meant by a vortex sheet in a fluid? What is the relation between the values of pressure on the two sides of a flat or plane vortex sheet? Does Bernoulli's principle apply to a fluid in which there is a vortex sheet?

81. Consider the water in a rotating bowl. Where is the velocity greatest? Where is the pressure greatest? Does Bernoulli's principle apply?

82. Consider the flow of water in a long horizontal pipe of uniform size. The velocity of the water is the same at both ends; is the pressure the same? If not, why not? Does Bernoulli's principle apply?

83. If one were to calculate the velocity of efflux of a gas from a high pressure tank, using Bernoulli's principle (supposing friction to be actually negligible) would the calculated velocity be too great or too small? Why?

84. Sketch carefully the stream lines in front of a flat plate set at an inclination to a fluid stream. Indicate the region of highest velocity and lowest pressure and the region of lowest velocity and highest pressure, and explain which way the plate tends to turn.

#### On pages 132-133:

Problem 139a, Ans. 234 gallons per hour.

" 139b, Ans. 0.368 cubic foot per second.

" 140, Ans. 6.94 feet in diameter.

" 141a, Ans. 3.07 inches in diameter.

" 141b, Ans. 0.192 "pound"; 14.4 foot-"pounds" per second.

#### On pages 137-139:

Problem 142, Ans. 14,800 "pounds" per square inch; 0.0005;  
 $2.96 \times 10^7$  "pounds" per square inch.

" 143, Ans. 1.36 inches.

" 145a, Ans. 68.1 pounds per cubic foot.

" 145b, Ans. 0.00053 inch.

#### On pages 142-143:

Problem 147, Ans. (a)  $R = 1071$  inches; (b) and (c) 41,700

"pounds" per square inch; (d) 188,000  
"pound"-inches.

Problem 148, Ans.  $2.986 \times 10^7$  "pounds" per square inch.

**On page 146:**

Problem 150, Ans.  $1.45 \times 10^7$  "pounds" per square inch.

" 151, Ans. 62.4 complete oscillations.

**On pages:**

Problem 155, Ans. Mean residual = 0.0025 inch (taken as  
probable error of each measurement);  
probable error of mean =  $\pm 0.00079$   
inch; probable error of area calculated  
from the mean =  $\pm 0.0012$  square inch.

" 156, Ans. Probable error of each measurement, as cal-  
culated by correct formula, =  $\pm 0.00235$ .

" 160, Ans. (a) 0.14 degree; (b) 35 degrees.

" 161, Ans. 8680 watts  $\pm 65$  watts.

" 162, Ans. (a) 26.50 ohms  $\pm 0.042$  ohm.  
(b) 6.287 ohms  $\pm 0.013$  ohm.

" 163, Ans.  $d\pi = \pm 0.0044$ .

" 164, Ans.  $\pm 0.001$  inch.

" 165, Ans. Permissible error in measured length is  
about  $\pm 0.058$  inch.

" 166, Ans.  $dL = \pm 7.8$  inches;  $dR = \pm 0.0067$  ohm.

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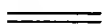


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